12.1 The Extrema of a Function

Question 1: What is the difference between a relative extremum and an absolute extremum?

Question 2: What is a critical point of a function?

Question 3: How do you find the relative extrema of a function?

Question 4: How do you find the absolute extrema of a function?

Many applications in business and economics require us to find either the highest point or the lowest point on a function. If we have a function that describes volume of a piece of luggage as a function of its dimensions, the highest point on the graph gives us the dimensions at which the volume is greatest. For a product's average cost function, the lowest point on the function gives us the production level that yields the lowest cost per unit.

In this section we'll find these points by learning how to find the bumps and dips on a function's graph as well as the very highest and lowest points on a graph.

Question 1: What is the difference between a relative extremum and an absolute extremum?

The high and low points on a graph are called the extrema of the function. An extremum that is higher or lower than any other points nearby is called a relative extremum. A relative extremum (the plural of extremum is extrema) that is higher than points nearby is called relative maximum. A relative extremum that is lower than points nearby is called a relative minimum.

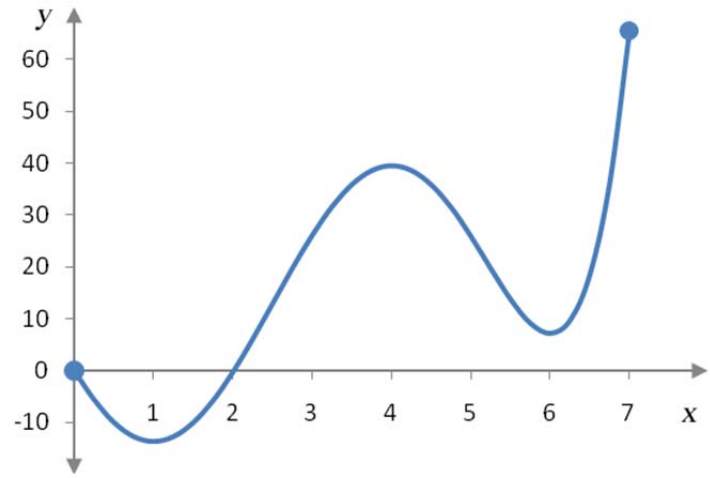
The point $(c, f(c))$ is a relative maximum of a function f if there exists an open interval (a,b) in the domain of f containing c such that $f(x) \le f(c)$ for all x in (a,b).

The point $(c, f(c))$ is a relative minimum of a function f if there exists an open interval (a,b) in the domain of f containing c such that $f(x) \ge f(c)$ for all x in (a,b).

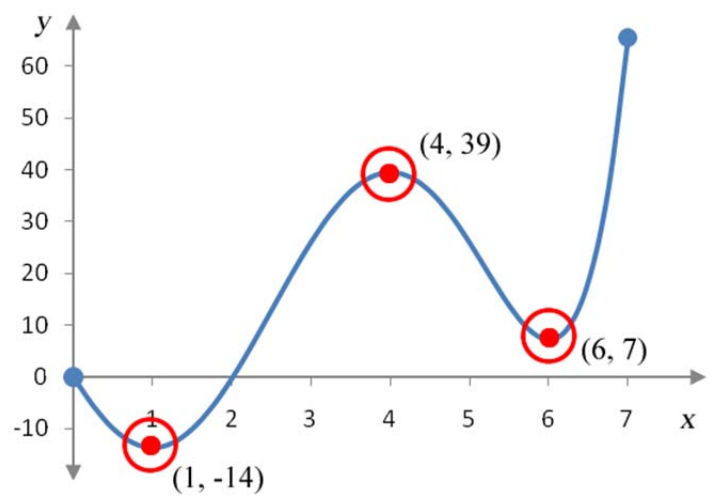
According to this definition, a relative extremum may not occur at the endpoint of an interval. Keep in mind that some textbooks define a relative extremum so that it may occur at the endpoints. The definition above is quite simple and will suffice for most business applications.

Example 1 Find the Relative Extrema

The function $p(x)$ is defined on the interval $[0,7]$ in the graph below. Find all relative extrema of $p(x)$.



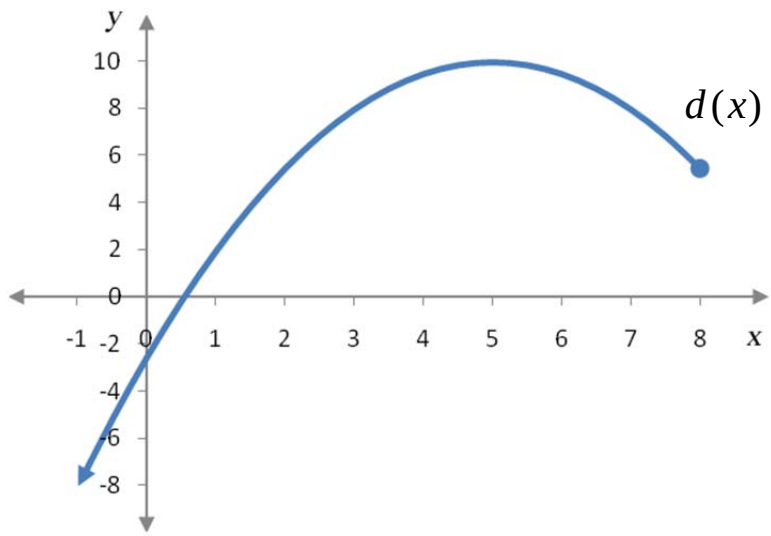
Solution The relative extrema are the highest and lowest points on the graph compared to the points nearby. We can check this by drawing a small circle around each potential relative extrema to illustrate the open interval.



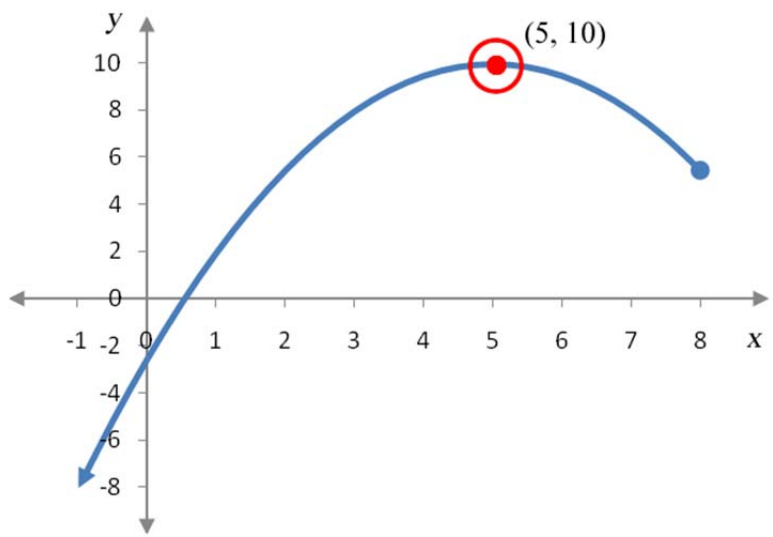
For x values in the small circles, the points $(1,-14)$ and $(6,7)$ are lower than any other points in the immediate neighborhood and are relative minima. The point $(4,39)$ is higher than any other points in the neighborhood of points indicated by the circle. This makes $(4,39)$ a relative maximum.

Example 2 Find the Relative Extrema

The function $d(x)$ is defined on the interval $(-\infty, 8]$ in the graph below. Find all relative extrema of $d(x)$.



Solution Since the point $(5,10)$ is higher than any other points in an open interval centered on $(5,10)$, the relative maximum occurs at that point. There are no troughs in the function so there are no relative minima on the function over this interval.



A function's absolute extremum occurs at the highest or lowest point on a function. The highest point on a function is called the absolute maximum and the lowest point on a function is called the absolute minimum.

Let f be a function defined on some interval and c be a number in that interval.

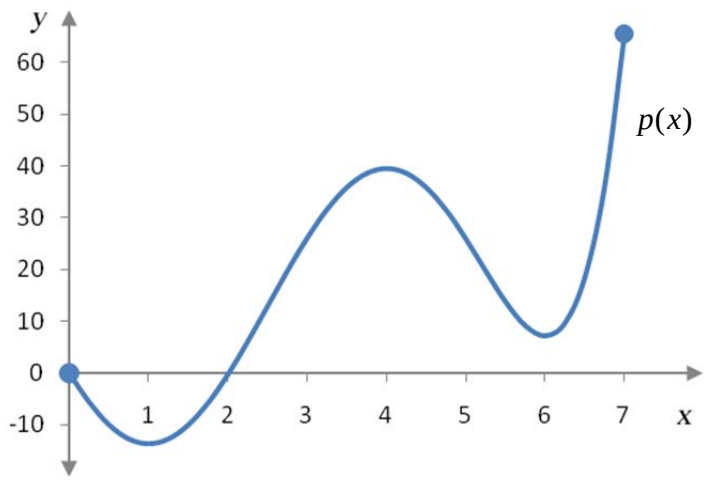
An absolute maximum of the function f occurs at $(c, f(c))$ if $f(x) \leq f(c)$ for every x value in the interval.

An absolute minimum of the function f occurs at $(c, f(c))$ if $f(x) \geq f(c)$ for every x value in the interval.

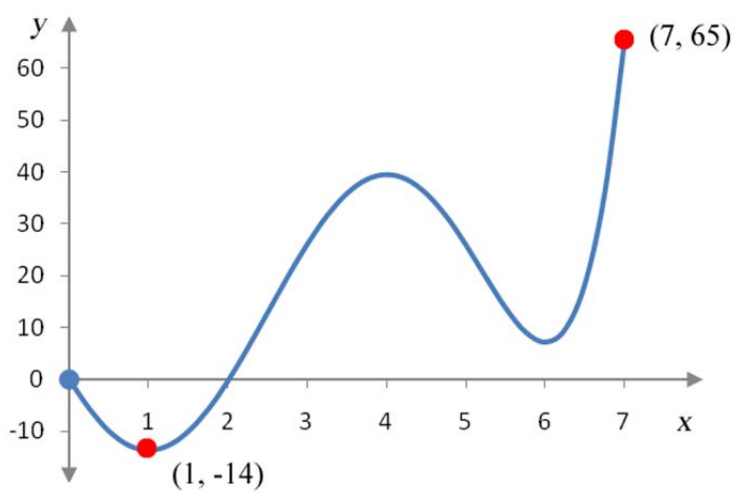
Since the interval may include the endpoints of the interval, absolute extrema may occur at the endpoints of the interval. This is an important distinction from relative extrema that are defined on open intervals.

Example 3 Find the Absolute Extrema

The function $p(x)$ is defined on the interval $[0,7]$ in the graph below. Find all absolute extrema of $p(x)$.



Solution The absolute extrema of $p(x)$ are the highest and lowest points on the graph over the interval $[0,7]$.

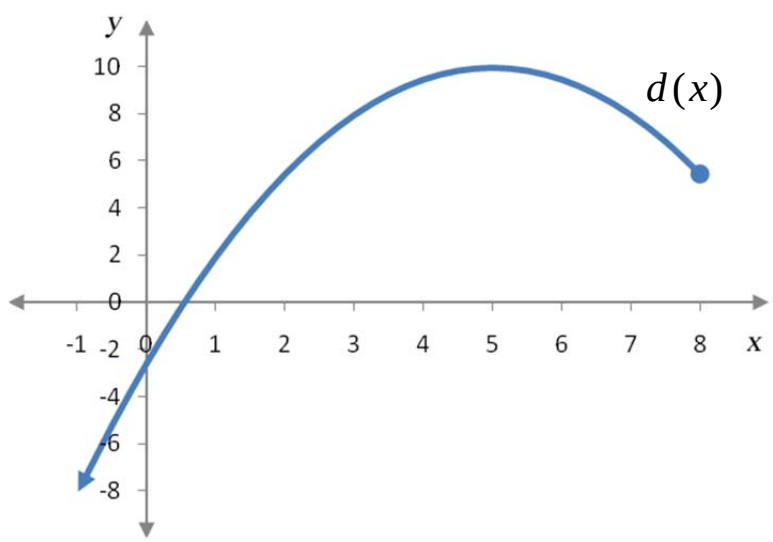


The absolute maximum occurs at the point $(7,65)$ since it is higher than any other point on the function over the interval $[0,7]$. The absolute

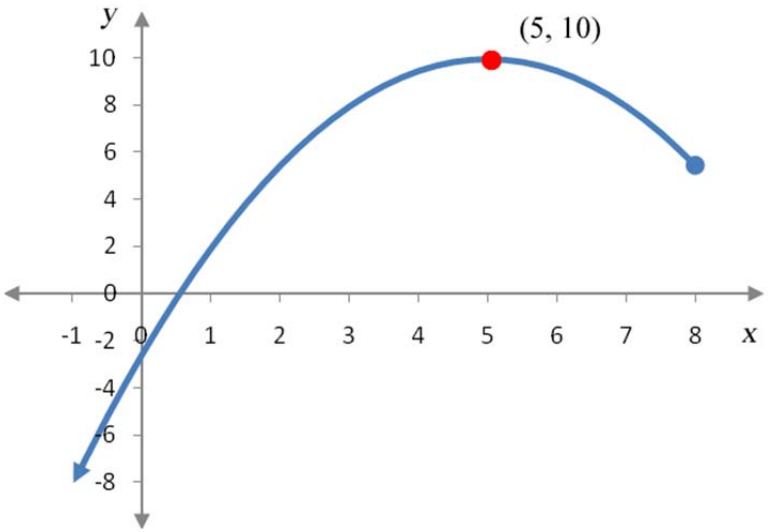
minimum occurs at the point $(1,-14)$ since it is lower than any other point on the function over the interval $[0,7]$.

Example 4 Find the Absolute Extrema

The function $d(x)$ is defined on the interval $(-\infty,8]$ in the graph below. Find all absolute extrema of $d(x)$.



Solution Unlike Example 3 this function is not defined on a closed interval. From the graph, it is apparent that the highest point on the graph is $(5,10)$ so the absolute maximum occurs at that point. The graph continues downward on the left side (as evidenced by the arrow on the left side of the graph) so there is no lowest point. Therefore, there is no absolute minimum.



In general, a function may or may not have relative extrema or absolute extrema. Compare the three graphs in Figure 1. Since there are no crests or troughs on these linear functions, we cannot find an open interval around any points in the intervals that are higher or lower than points nearby. This indicates that there are no relative extrema on any of the graphs.

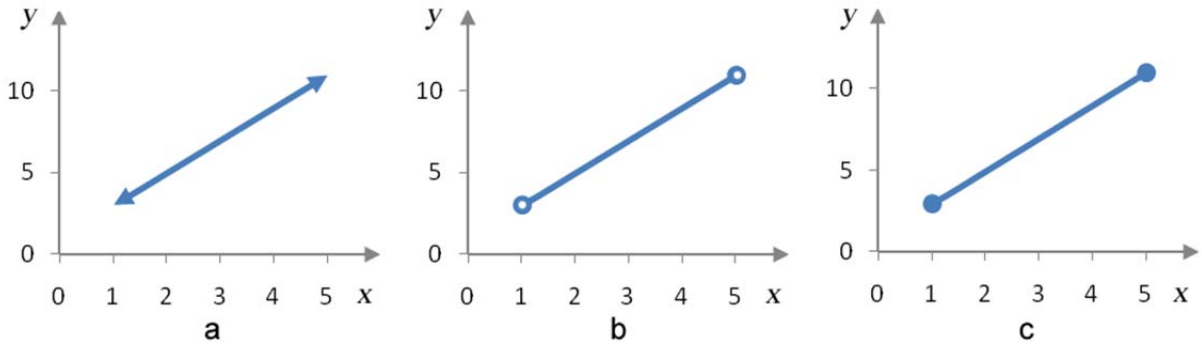


Figure 1 – A linear function defined on three different intervals. In (a), the function is defined for all real numbers. In (b), the function is defined on the open interval (1, 7). The function in (c) is defined on a closed interval [1,5].

In Figure 1a, there are no absolute extrema since the graph continues indefinitely up on the right side and indefinitely down on the left side. You might think that the lack of absolute extrema would be remedied by cutting the graph off at some values as in

Figure 1b. The extreme values should be at the endpoints of the interval, but since the interval is an open interval, the endpoints are not included on the function.

In Figure 1b, we might try to say that the absolute maximum and minimum are very close to the ends of the interval, at $x = 4.9$ and $x = 1.1$. If this were the case, these x values would correspond to ordered pairs that are higher or lower than any other ordered pairs over the open interval. By inspecting the graph closely, we can see that the ordered pair corresponding to $x = 4.99$ is higher than the ordered pair for $x = 4.9$ and the ordered pair corresponding to $x = 1.01$ is lower than the ordered pair for $x = 1.1$. Since we can always move a little bit closer to the ends of the open interval, we'll never be able to find a point we can call the highest or lowest. This can only happen for this function if it is defined on a closed interval as in Figure 1c.

To guarantee an absolute maximum and absolute minimum on a function, it needs more than simply being defined on a closed interval .The function in **Error! Reference source not found.** has an absolute minimum at $(0,1)$, but no absolute maximum.

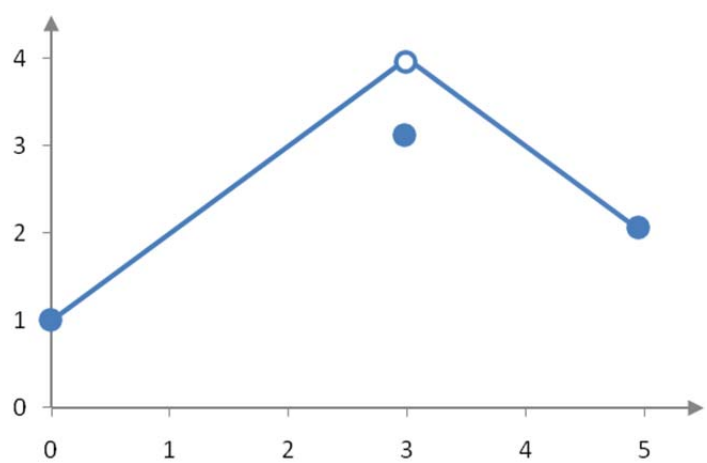


Figure 2 - A discontinuous function defined on a closed interval [0,5].

The absolute maximum does not exist because this is not a continuous function. Continuous functions are functions whose graphs can be drawn by a pencil without lifting the pencil from the page. The discontinuity at $x = 3$ occurs because we would have to lift the pencil to draw the point there.

There appears to be a highest point on the graph near $(3,4)$, but where? We could say it is at $x=2.9$ or $x=3.1$, but points closer to $x=3$ like $x=2.99$ are higher. In fact, the closer we get to $x=3$, the higher the graph is. Since we would never be able to name a point that is highest, there is no absolute maximum.

A closely related continuous function is graphed in Figure 3. The absolute maximum of this function occurs at $(3,4)$ since it is the highest point on the function over the closed interval $[0,5]$. The absolute minimum of the function occurs at $(0,1)$ since this point is the lowest point over the closed interval $[0,5]$.

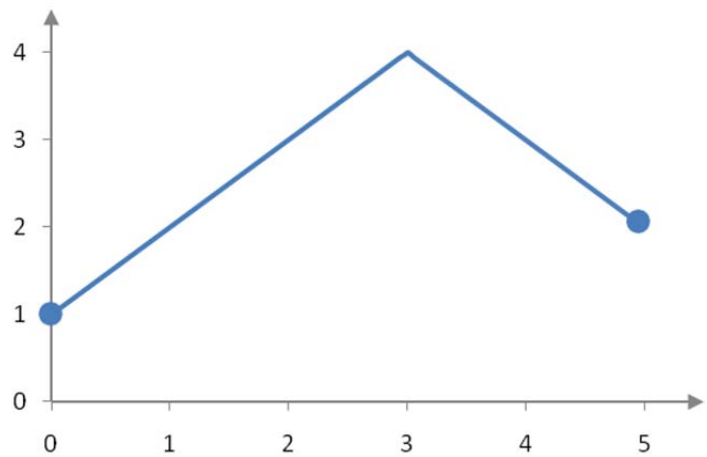


Figure 3 - A continuous function defined on the interval [0,5].

The Extreme Value Theorem gives the conditions under which the absolute extrema are guaranteed.

Extreme Value Theorem

A function f that is continuous on a closed interval is guaranteed to have both an absolute maximum and an absolute minimum.

In the rest of this section, we'll learn how to find absolute extrema and relative extrema of a function using the derivative of the function.

Question 2: What is a critical point of a function?

The graph of a function may rise and fall resulting in relative and absolute extrema. For instance, if a continuous function falls and then rises as we move from left to right on its graph then we can deduce that the function must have a relative minimum.

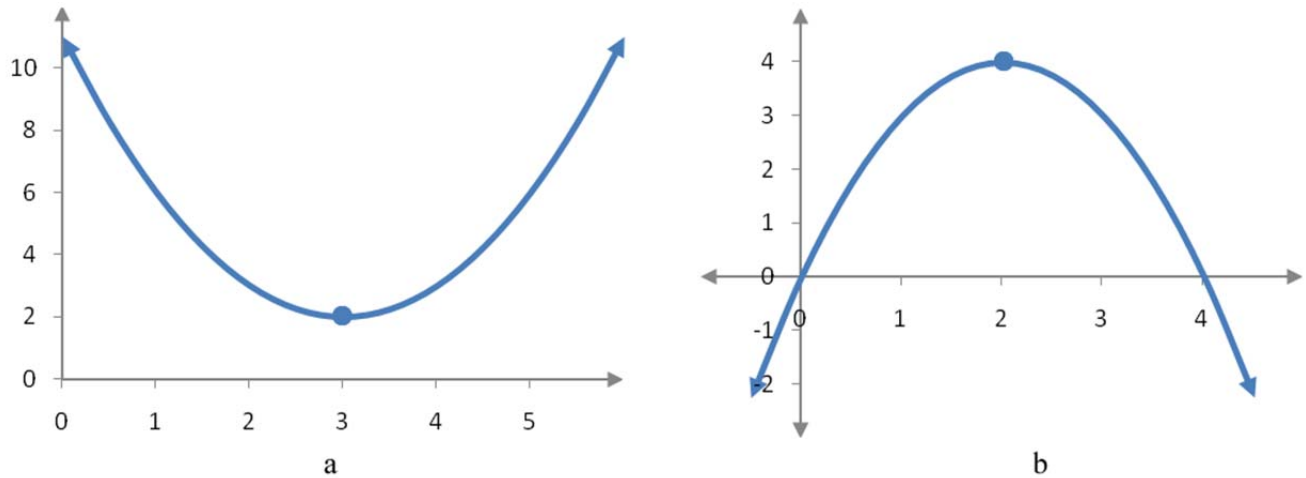


Figure 4 – When a function falls and then rises, we get a relative minimum as in (a). In (b), the function rises and then falls resulting in a relative maximum.

A continuous function that rises on the left and then drops on the right has a relative maximum. The terms increasing and decreasing are used to describe when a function is rising or falling as we move from the left to the right on the graph of a function.

Suppose x_1 and x_2 are two numbers in an interval over which a function f is defined.

f is increasing on the interval if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.

f is decreasing on the interval if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

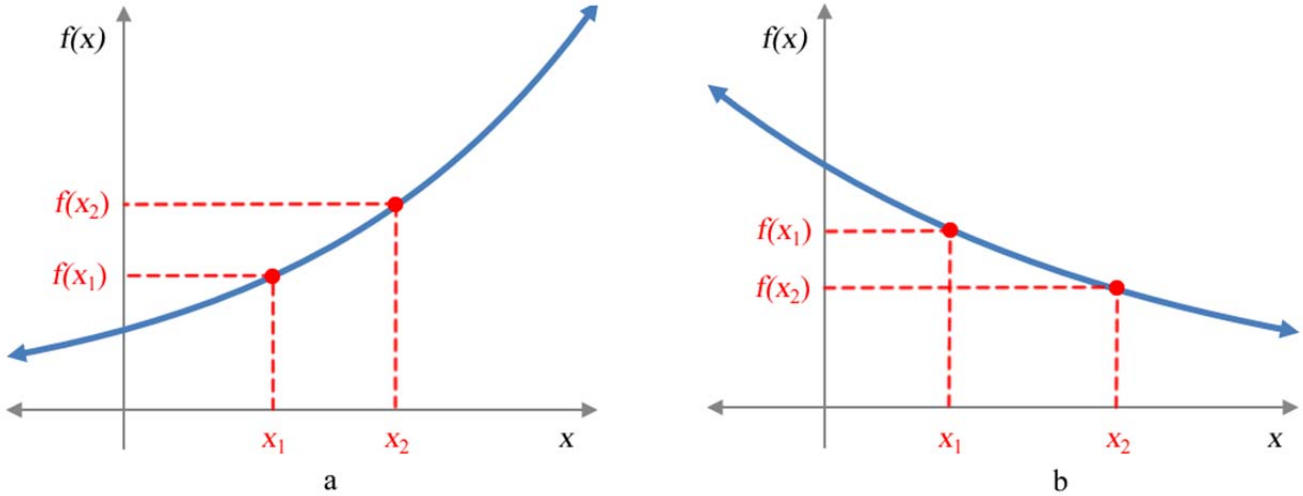


Figure 5 – In (a), function values on the right are higher than function values on the left so the function is increasing. In (b), function values on the right are smaller than function values on the left so the function is decreasing.

Critical points are ordered pairs on the graph where the function can potentially change from increasing to decreasing or decreasing to increasing. This means that we will be able to use these critical numbers to find the relative extrema of a function later in this section.

Let f be a continuous function defined on an open interval containing a number c. The number c is critical value (or critical number) if $f'(c)=0$ or $f'(c)$ is undefined. A critical point on the graph of f has the form $(c, f(c))$.

When necessary, we'll use the term stationary value to indicate critical values where the derivative is equal to zero and the term singular value to indicate critical values where the derivative is undefined.

We can test the derivative of a function at test values between the critical values to find the sign of the slope of the tangent line at the test values. This allows us to deduce whether the function is increasing or decreasing between the critical values.

Test for Determining the Intervals Where a Function is Increasing or Decreasing

Suppose a function f is defined on an open interval and that we are able to compute the derivative of the function at each point in the open interval.

If $f'(x) > 0$ at every x value in the interval, then f is increasing on the interval.

If $f'(x) < 0$ at every x value in the interval, then f is decreasing on the interval.

If $f'(x) = 0$ at every x value in the interval, then f is constant on the interval.

Using this test, we can write down a strategy for finding all intervals where a function is increasing or decreasing.

Strategy for Tracking the Sign of the Derivative

1. Take the derivative of the function and use it to find all of the critical values. Below a number line, label these values. Above the number line, write = 0 to indicate critical values where the derivative is zero or write und to indicate critical values where the derivative is undefined. The open intervals between these values are where we will determine the sign of the derivative.

2. Pick a value in each of the open intervals between the critical values. Substitute these values in $f'(x)$ to determine

whether the derivative is positive or negative at these values. Label the number line + or – to indicate whether the derivative is positive or negative.

3. The function is increasing over the intervals where $f'(x) > 0$ and decreasing over the intervals where $f'(x) < 0$.

Example 5 Find Where a Function is Increasing and Decreasing

Let $f(x) = x^4 - 18x^2$. Use the derivative of this function to answer the questions below.

a. Find the derivative of $f(x)$.

Solution The derivative is found by applying several derivative rules,

$$f'(x) = \frac{d}{dx}\left[x^4\right] - 18\frac{d}{dx}\left[x^2\right]$$

Use Sum / Difference Rule and the Product with a Constant Rule for derivatives

$$= 4x^3 - 18(2x)$$

Use the Power Rule for Derivatives

$$= 4x^3 - 36x$$

So the derivative of $f(x)$ is $f'(x) = 4x^3 - 36x$.

b. Use the derivative to find the critical values of $f(x)$.

Solution The critical values of $f(x)$ are where $f'(x)$ is equal to zero or undefined. Since $f'(x)$ is a polynomial, it is defined everywhere. All critical values are where $f'(x) = 0$. Set the derivative equal to zero and solve for x to find these points,

$$4x^3 - 36x = 0$$ Set the derivative equal to zero

$$4x\left(x^2 - 9\right) = 0$$ Factor the greatest common factor, $4x$

$$4x(x-3)(x+3) = 0$$ Factor the difference of squares $x^2 - 9$

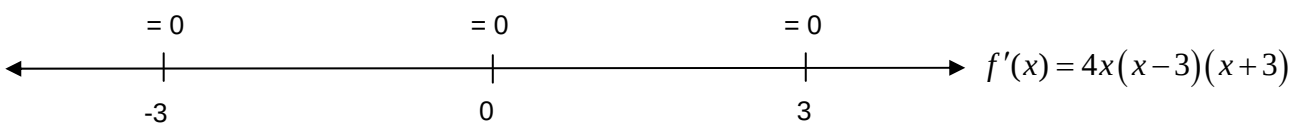
$$4x = 0 \quad x - 3 = 0 \quad x + 3 = 0$$ Set each factor equal to zero

$$x = 0 \qquad x = 3 \qquad x = -3$$ Solve each equation for x

The critical values are $x = 0, 3$ and -3.

c. Find the intervals where $f(x)$ is increasing and the intervals where $f(x)$ is decreasing.

Solution A function is increasing when the derivative of the function is positive and decreasing when the derivative of the function is negative. Make a number line and place the critical numbers on the number line. Above each critical number place the value of the derivative, either equal to zero or undefined.



These three critical numbers divide the number line into four sections: $(-\infty, -3)$, $(-3, 0)$, $(0, 3)$ and $(3, \infty)$. In each of these intervals, the graph is either increasing or decreasing. The critical numbers are where the graph could potentially change from increasing to decreasing of vice versa.

To decide where the graph is increasing or decreasing, we need to pick a test point from each interval and use it to determine the sign of the derivative at the test point. Although we can test the point in the derivative $f'(x) = 4x^3 - 36x$, it is easier to use the factored form of the derivative

$$f'(x) = 4x(x-3)(x+3)$$

For instance, the test point $x = -4$ is in the interval $(-\infty, -3)$. If we substitute this value into the derivative we get

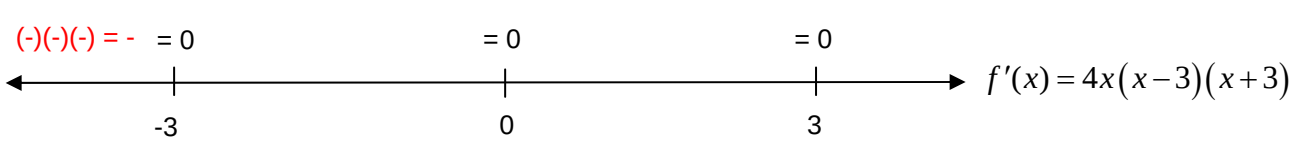
$$\begin{aligned} f'(-4) &= 4(-4)(-4-3)(-4+3) \\ &= (-16)(-7)(-1) \\ &= -112 \end{aligned}$$

This tells us the function's graph is decreasing in the interval $(-\infty, -3)$. To find where the graph is increasing, we need to test the derivative in each of the intervals $(-\infty, -3)$, $(-3, 0)$, $(0, 3)$ and $(3, \infty)$.

To simplify the testing of the derivative, we need to realize that we only need to know the sign of the derivative, not the actual value. If we substitute the value $x = -4$ into the derivative and note the sign of each factor, we can quickly determine the sign of the product of the factors:

$$f'(-4) = \underbrace{4(-4)}_{\text{negative}}\underbrace{(-4-3)}_{\text{negative}}\underbrace{(-4+3)}_{\text{negative}}$$

A simple way to symbolize this result is to label the signs of the factors on the number line.

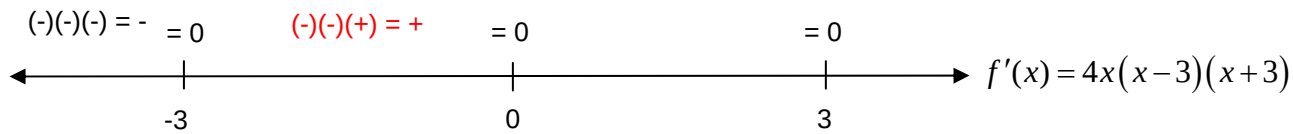


The product of three negative factors is negative so we know the function is decreasing in the interval $(-\infty,-3)$.

Let's examine the test point $x=-1$ in the interval $(-3,0)$. When we examine the signs of the factors in the derivative at this value,

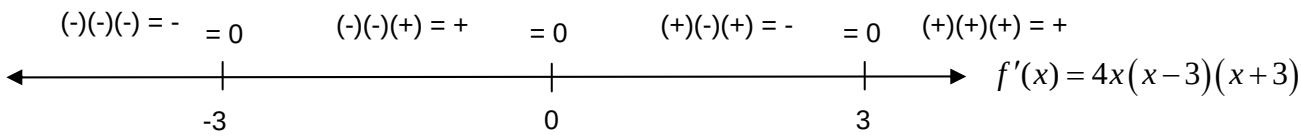
$$f'(-1)=\underbrace{4(-1)}_{\text{negative}}\underbrace{(-1-3)}_{\text{negative}}\underbrace{(-1+3)}_{\text{positive}}$$

Label these signs above the interval on the number line:

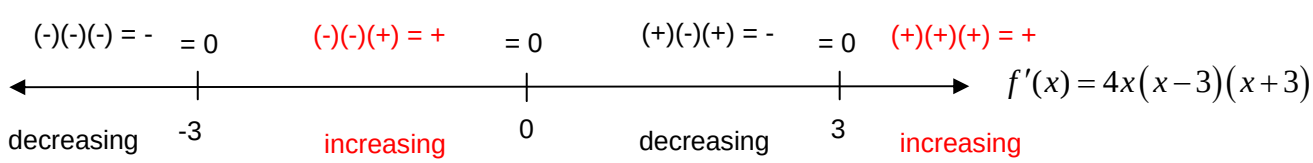


The product of two negative numbers and a positive number is positive so the graph is increasing in the interval $(-3,0)$.

If we continue testing and labeling the first derivative number line, we see exactly where the first derivative is positive and negative.

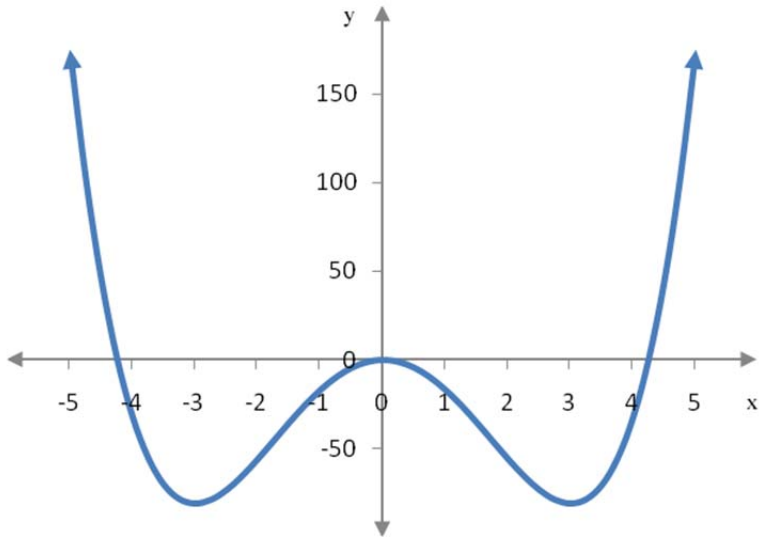


Based on this number line, the derivative is positive in the intervals $(-3,0)$ and $(3,\infty)$. This means the graph of the function is increasing in the intervals $(-3,0)$ and $(3,\infty)$.



With the number line labeled like this we can also observe that the graph is decreasing in the intervals $(-\infty,-3)$ and $(0,3)$.

We can check this number line by examining the graph of the function.



From the graph we can verify that the function increases on $(-3,0)$ and $(3,\infty)$. However, we should be cautious when using the graph as our only evidence of where the graph is increasing. We can only read approximate values from the graph. The derivative allows us to find the intervals exactly.

Question 3: How do you find the relative extrema of a function?

The strategy for tracking the sign of the derivative is useful for more than determining where a function is increasing or decreasing. It is also useful for locating the relative extrema of a function. At a relative extrema, a function changes from increasing to decreasing or decreasing to increasing. The number lines in the previous question allow us to see these changes by observing changes in the sign of the derivative of a function.

When the derivative of a function changes from positive to negative, we know the function changes from increasing to decreasing. As long as the function is defined at the critical value where the change occurs, the critical point must be a relative maximum. If the derivative of a function changes from negative to positive, we know the function changes from decreasing to increasing. In this case, the critical point is a relative minimum as long as the function is defined there. If the derivative does not change sign at a critical value, there is no relative extrema at the corresponding critical point.

The First Derivative Test summarizes these observations and helps us to locate relative extrema on a function.

First Derivative Test

Let f be a non-constant function that is defined at a critical value $x = c$.

If f' changes from positive to negative at $x = c$, then a relative maximum occurs at the critical point $(c, f(c))$.

If f' changes from negative to positive at $x = c$, then a relative minimum occurs at the critical point $(c, f(c))$.

If f' does not change sign at $x = c$, then there is no relative extrema at the corresponding critical point.

Example 6 Find the Relative Extrema of a Function

Find the location of the relative extrema of the function

$$f(x) = 4x^3 - 21x^2 + 18x + 5$$

Solution The first derivative test requires us to construct a number line for the derivative so that we can identify where the graph is increasing and decreasing. Using the rules for derivatives, the first derivative of the function $f(x)$ is

$$f'(x) = 4\frac{d}{dx}\left[x^3\right] - 21\frac{d}{dx}\left[x^2\right] + 18\frac{d}{dx}[x] + \frac{d}{dx}[5]$$

Use Sum / Difference Rule and the Product with a Constant Rule.

$$= 4\left(3x^2\right) - 21(2x) + 18(1) + 0$$

Use the Power Rule for Derivatives and the fact that the derivative of a constant is zero

So the derivative is $f'(x) = 12x^2 - 42x + 18$.

We need to use this derivative to find the critical values. Set the derivative, $f'(x) = 12x^2 - 42x + 18$, equal to zero to find those values.

$$12x^2 - 42x + 18 = 0$$

Set the derivative equal to zero

$$6\left(2x^2 - 7x + 3\right) = 0$$

Factor the greatest common factor from each term

$$6(2x-1)(x-3) = 0$$

Factor the trinomial

$$2x - 1 = 0 \qquad x - 3 = 0$$

$$2x = 1 \qquad x = 3$$

$$x = \tfrac{1}{2}$$

To find where the product is equal to zero, set each factor equal to zero and solve for the variable

In general, critical values may also come from x values where the derivative is undefined. Since $f'(x)$ is a polynomial, it is defined everywhere so the derivative is defined everywhere.

Although this derivative could be factored to find the critical values, most quadratic derivatives are not factorable. In this case, the quadratic equation yielding the critical values can be solved using the quadratic formula. This strategy would yield the same critical values as factoring:

$$\underbrace{12}_{a}x^2 \underbrace{-42}_{b}x + \underbrace{18}_{c} = 0$$

Identify a, b, and c for the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ and put in the values

$$x = \frac{-(-42) \pm \sqrt{(-42)^2 - 4(12)(18)}}{2(12)}$$

$$= \frac{42 \pm \sqrt{900}}{24}$$

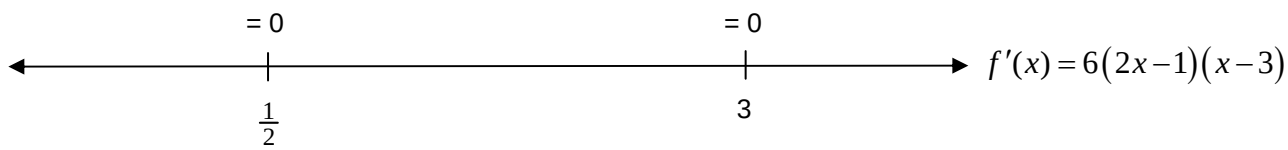
Simplify the numerator and denominator

$$= \frac{42 \pm 30}{24}$$

$$= 3, \tfrac{1}{2}$$

To find the critical values with more complicated derivatives, we may need to solve the equation using a graph. The solution to the equation

$12x^2 - 42x + 18 = 0$ can also be found by locating the x intercepts on the derivative $f'(x)$.

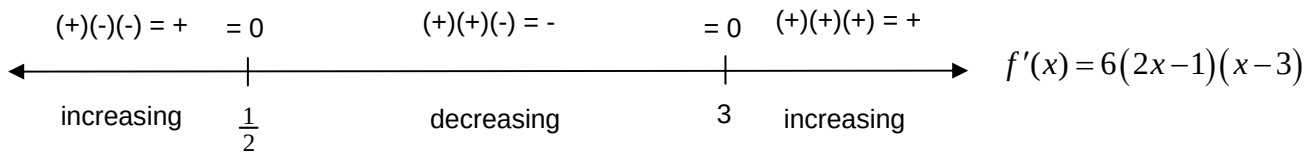


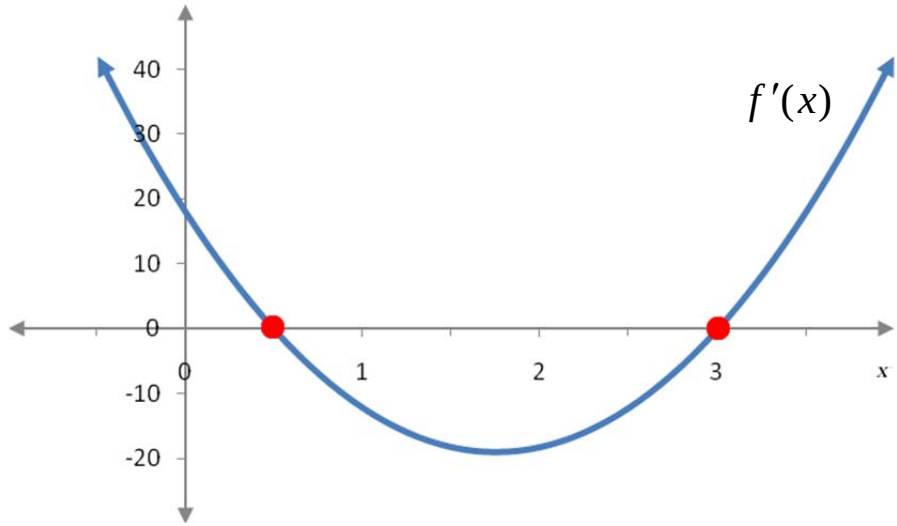
Like factoring or the quadratic formula, the critical values are located at $x = \frac{1}{2}$ and $x = 3$. All three strategies yield the same critical values. Keep in mind that a graph will give approximate values while factoring or the quadratic formula yield exact values.

If the derivative is not factorable, linear, or quadratic, another method will need to be used to determine where the derivative is equal to zero. Even though a graph of the derivative only gives an estimate of the critical values, it may be the only way to find the critical values if the derivative is complicated.

With the critical values in hand, label them on a number line so that we are able to apply the first derivative test.

If we select a test point in each interval and determine the sign of each factor, we can complete the number line and track the sign of the derivative.





When a continuous function changes from increasing to decreasing, we have a relative maximum at the critical value. When a continuous function changes from decreasing to increasing, we have a relative minimum at the critical value. In this case, the relative maximum is located at $x = \frac{1}{2}$ and the relative minimum is located at $x = 3$.

To find the ordered pairs for the relative extrema, we need to substitute the critical values into the original function $f(x)$ to find the corresponding y values:

$$\text{Relative Maximum:} \quad f\left(\tfrac{1}{2}\right) = 4\left(\tfrac{1}{2}\right)^3 - 21\left(\tfrac{1}{2}\right)^2 + 18\left(\tfrac{1}{2}\right) + 5 = \tfrac{37}{4}$$

$$\text{Relative Minimum:} \quad f(3) = 4(3)^3 - 21(3)^2 + 18(3) + 5 = -22$$

The relative maximum is located at $\left(\frac{1}{2}, \frac{37}{4}\right)$ and the relative minimum is located at $(3, -22)$.

Example 7 Find the Relative Maximum

Find the location of the relative maximum of the function

$$g(x) = \frac{x}{e^x}$$

Solution The derivative of $g(x)$ is found with the Quotient Rule for Derivatives with

$$u = x \qquad v = e^x$$

$$u' = 1 \qquad v' = e^x$$

Put these expressions into the Quotient Rule:

$$g'(x) = \frac{e^x(1) - x(e^x)}{(e^x)^2}$$

The Quotient Rule is $\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}$

To make it easier to find the critical values, simplify the derivative.

$$\frac{e^x(1) - x(e^x)}{(e^x)^2} = \frac{e^x(1-x)}{(e^x)^2}$$

Factor e^x from the numerator

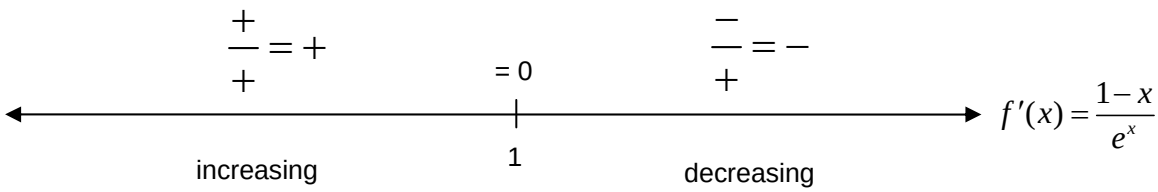
$$= \frac{1-x}{e^x}$$

Reduce the common e^x factor in the numerator and denominator

Now we can find the critical values by determining where this fraction is equal to zero or undefined.

Any fraction is equal to zero where the numerator is equal to zero. In this case, this is where $1 - x = 0$ or $x = 1$. Fractions are undefined where the denominator is equal to zero. The denominator for this fraction is e^x and is always positive so there are no x values where the fraction is undefined.

To apply the first derivative test, label a number line with this critical value and test the first derivative on either side of the critical value:



Since this function is continuous and the derivative changes from increasing to decreasing at $x = 1$, the critical point is a relative maximum. The y value for the relative maximum comes from the function $g(x) = \frac{x}{e^x}$ and is $g(1) = \frac{1}{e^1}$. The ordered pair for the relative maximum is $\left(1, \frac{1}{e}\right)$.

Example 8 Find the Minimum Average Cost

The total daily cost to produce Q units of a product is given by the function

$$C(Q) = 0.005Q^2 + 10Q + 1000 \text{ dollars}$$

The average cost function $\overline{C}(Q)$ is found by dividing the total daily cost function $C(Q)$ by the quantity Q.

a. Find the average cost function $\overline{C}(Q)$.

Solution The average cost function is defined by $\overline{C}(Q) = \frac{C(Q)}{Q}$.

Substitute the total daily cost function into the numerator of this fraction to yield

$$\overline{C}(Q) = \frac{0.005Q^2 + 10Q + 1000}{Q}$$

b. Find the quantity that yields the minimum average cost.

Solution The minimum average cost is found by locating the relative minimum of the average cost function. To use the first derivative test to find this relative minimum, we need to take the derivative of $\overline{C}(Q)$ using the Quotient Rule for Derivatives. The numerator, denominator and their derivatives are

$$u = 0.005Q^2 + 10Q + 1000 \qquad v = Q$$

$$u' = 0.010Q + 10 \qquad v' = 1$$

The derivative of the average cost function is

$$\overline{C}'(Q) = \frac{\overbrace{Q(0.010Q+10)}^{vu'} - \overbrace{(0.005Q^2 + 10Q + 1000)\cdot 1}^{uv'}}{\underbrace{Q^2}_{v^2}}$$

Since we need to use the derivative to find the critical values, we simplify the derivative as much as possible:

$$\overline{C}'(Q) = \frac{0.010Q^2 + 10Q - 0.005Q^2 - 10Q - 1000}{Q^2}$$

Simplify the numerator by carrying out the multiplication and subtraction

$$= \frac{0.005Q^2 - 1000}{Q^2}$$

Combine like terms

Any fraction is equal to zero when the numerator is equal to zero and undefined where the denominator is equal to zero.

Set the numerator equal to 0

$$0.005Q^2 - 1000 = 0$$

$$0.005Q^2 = 1000$$

$$Q^2 = 200000$$

$$Q = \pm\sqrt{200000}$$

$$Q \approx \pm 447$$

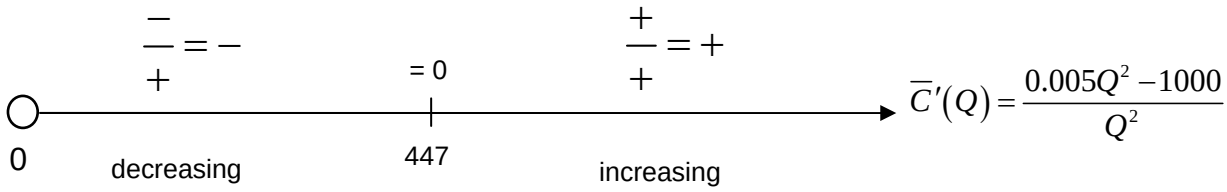
Set the denominator equal to 0

$$Q^2 = 0$$

$$Q = 0$$

Since quantities produced must be positive, only $Q \approx 447$ is a reasonable critical value for this function.

The number line for this function only includes positive quantities. Testing on either side of the critical numbers yields the behavior of $\overline{C}'$.



Since the function decreases on the left side of the critical value and increases on the right side of the critical value, the quantity at approximately 447 units is a relative minimum.

The minimum average cost is obtained from

$\overline{C}(Q) = \dfrac{0.005Q^2 + 10Q + 1000}{Q}$ and is calculated as

$$\overline{C}\left(\sqrt{200000}\right) = \frac{0.005\left(\sqrt{200000}\right)^2 + 10\left(\sqrt{200000}\right) + 1000}{\sqrt{200000}}$$

$$\approx 14.47$$

Since the total daily cost is in dollars and we are dividing by the number of units to get the average cost, the units on the average cost are dollars per unit. This means that a production level of about 447 units gives the lowest average cost of 14.47 dollars per unit.

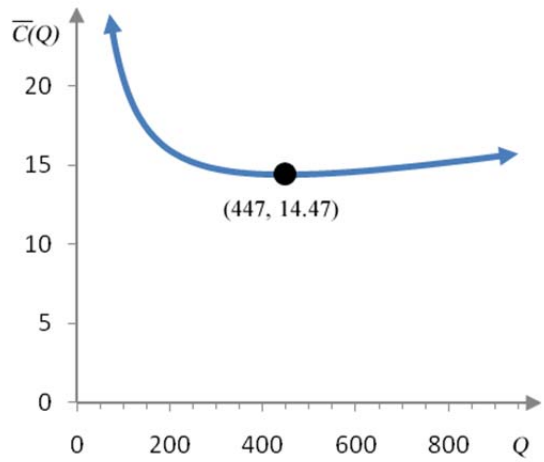


Figure 6 - The relative minimum for the average cost function.

Question 4: How do you find the absolute extrema of a function?

The absolute extrema of a function is the highest or lowest point over which a function is defined. In general, a function may or may not have an absolute maximum or absolute minimum. However, under certain conditions a function will automatically have absolute extrema. The Extreme Value Theorem guarantees that a continuous function defined over a closed interval will have both an absolute maximum and an absolute minimum. These extrema will occur at the critical values or at the end points on the closed interval.

To find the absolute extrema from these possibilities, we must determine which of these values yields the highest and lowest values of the function. This is done by testing each critical value and the endpoints in the function. The highest value of the function is the absolute maximum and the lowest value is the absolute minimum.

Strategy for Finding Absolute Extrema

To find the absolute extrema on a continuous function f defined over a closed interval,

1. Find all critical values for the function f on the open interval.

2. Evaluate each critical value in the function f.

3. Evaluate each endpoint of the closed interval in the function f.

4. The largest function value from steps 2 and 3 is the absolute maximum and the smallest function value from steps 2 and 3 is the absolute minimum.

Notice that the absolute extrema are the function values, not the critical values or endpoints. However, the absolute extrema occur at points on the graph given by an ordered pair.

Example 9 Find the Absolute Extrema of a Function

Find the absolute maximum and absolute minimum of the function

$$f(x) = 8x^2 - \tfrac{4}{3}x^3$$

on the closed interval $[1,6]$.

Solution This function is a polynomial so it is continuous not only on the closed interval $[1,6]$, but everywhere. The absolute extrema of a continous function over a closed interval will occur at a critical value in the interval or at the endpoints. We can located the critical values of this function from the derivative,

$$f'(x) = 8\frac{d}{dx}\left[x^2\right] - \frac{4}{3}\frac{d}{dx}\left[x^3\right]$$ Apply the Sum / Difference Rule and the Product with a Constant Rule for Derivatives

$$= 8(2x) - \frac{4}{3}(3x^2)$$ Use the Power Rule for Derivatives

$$= 16x - 4x^2$$ Multiply the constants in each term

Since $f(x)$ is a polynomial and defined everywhere, the only critical values are due to where the derivative is equal to zero. Set $f'(x)$ equal to zero and solve for x:

$$0 = 16x - 4x^2$$ Set the derivative equal to zero

$$0 = 4x(4 - x)$$ Factor the greatest common factor, $4x$, from each term

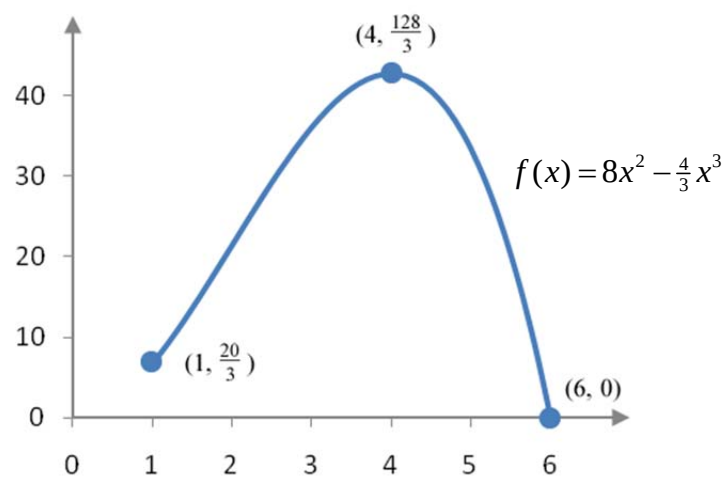
$$4x = 0 \quad 4 - x = 0$$ Set each factor equal to zero and solve for x

$$x = 0 \qquad 4 = x$$

One of these critical values, $x=0$ is not in the interval $[1,6]$ so we can ignore it.

The other critical point at $x=4$ and endpoints at $x=1$ and $x=6$ are substituted into $f(x)=8x^2-\frac{4}{3}x^3$ in order to find the highest and lowest points on the graph.

x Values	Function Value	Ordered Pair
$x=1$	$f(1)=8(1)^2-\frac{4}{3}(1)^3=\frac{20}{3}$	$\left(1,\frac{20}{3}\right)$
$x=4$	$f(4)=8(4)^2-\frac{4}{3}(4)^3=\frac{128}{3}$	$\left(4,\frac{128}{3}\right)$
$x=6$	$f(6)=8(6)^2-\frac{4}{3}(6)^3=0$	$(6,0)$



The absolute maximum occurs at $\left(4,\frac{128}{3}\right)$ since it has the largest y value.

The absolute minimum occurs at $(6,0)$ since it has the smallest y value.

Example 10 Find the Absolute Extrema of a Function

Find the absolute maximum and absolute minimum of the function

$$f(x) = \frac{\ln x}{x}$$

on the closed interval $[2,10]$.

Solution The natural logarithm is continuous over $[2,10]$ and the denominator is equal to zero outside of this interval. This means the quotient is continuous over $[2,10]$. Therefore the absolute extrema are located at the critical values or the endpoints of the interval. To find the critical values, calculate the derivative with the quotient rule.

The numerator, denominator and their derivatives are

$$u = \ln x \qquad v = x$$

$$u' = \tfrac{1}{x} \qquad v' = 1$$

The derivative is

$$f'(x) = \frac{x\left(\frac{1}{x}\right) - \ln x(1)}{x^2}$$

Apply the Quotient Rule $\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}$

$$= \frac{1 - \ln x}{x^2}$$

Simplify each term in the numerator

The critical values of a function are where the derivative is equal to zero or undefined. For a fraction like this one, the derivative is undefined where the denominator is equal to zero. This occurs when $x = 0$. However, this value is outside of the interval $[2,10]$ so it can be ignored.

To find where the derivative is equal to zero, set the numerator of $f'(x) = \frac{1 - \ln(x)}{x^2}$ equal to zero and solve for x:

$1-\ln(x)=0$	Set $1-\ln(x)$ equal to zero
$-\ln(x)=-1$	Subtract 1 from both sides
$\ln(x)=1$	Divide both sides by -1
$x=e^1$	Convert to exponential form

This critical value is at $x=e^1$. This value is approximately 2.72 and is in the interval $[2,10]$.

To find the absolute extrema, we need to substitute the critical value at $x=e^1$ and the endpoints of the interval at $x=2,10$ into $f(x)$.

Using the function $f(x)=\dfrac{\ln(x)}{x}$, we get the location of each ordered pair at the critical number and endpoints.

x Values	Function Value	Ordered Pair
$x=2$	$f(2)=\dfrac{\ln(2)}{2}\approx 0.35$	$(2,0.35)$
$x=e^1$	$f(e^1)=\dfrac{\ln(e^1)}{e^1}\approx 0.37$	$(e^1,0.37)$
$x=10$	$f(10)=\dfrac{\ln(10)}{10}=0.23$	$(10,0.23)$

The absolute maximum occurs at approximately $(e^1,0.37)$ and the absolute minimum occurs at approximately $(10,0.23)$.

Example 11 Find the Absolute Extrema of a Function

Verizon Wireless charges each customer a monthly charge for service on its wireless network. This charge is recorded as service revenue in corporate reports. The average annual service revenue per customer (in dollars) at Verizon Wireless from 2004 to 2009 can be modeled by the function

$$\overline{R}(t) = -2.33t^3 + 46.25t^2 - 285.22t + 1107.80$$

where t is the number of years since 2000.

(Source Verizon Annual Reports 2004 through 2009)

a. Over the period 2004 to 2009, when was the the average annual service revenue per customer highest?

Solution Since this function is defined from 2004 to 2009, the variable t is defined on the closed interval $[4,9]$. The average annual service revenue per customer is highest at the absolute maximum on this interval.

The derivative of $\overline{R}(t) = -2.33t^3 + 46.25t^2 - 285.22t + 1107.80$ is found using the basic rules for taking derivatives and the Power Rule for Derivatives:

$$\begin{aligned}\overline{R}'(t) &= -2.33\frac{d}{dt}\left[t^3\right] + 46.25\frac{d}{dt}\left[t^2\right] - 285.22\frac{d}{dt}[t] + \frac{d}{dt}[1107.80] \\ &= -2.33\left(3t^2\right) + 46.25(2t) - 285.22(1) + 0 \\ &= -6.99t^2 + 92.50t - 285.22\end{aligned}$$

Critical values for this polynomial are found by setting this derivative equal to zero and solving for the variable:

$$\underbrace{-6.99}_{a}t^2 + \underbrace{92.50}_{b}t \underbrace{-285.22}_{c} = 0$$

Set the derivative equal to zero and identify the coefficients

$$t = \frac{-92.50 \pm \sqrt{92.50^2 - 4(-6.99)(-285.22)}}{2(-6.99)}$$

$$\approx 4.89, 8.34$$

Solve the equation using the quadratic formula $t = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ and simplify

Both of these critical values are in the interval $[4,9]$.

Evaluate $\bar{R}(t) = -2.33t^3 + 46.25t^2 - 285.22t + 1107.80$ at the two critical values and the endpoints to find the absolute maximum.

x Values	Function Value	Ordered Pair
$t = 4$	$\bar{R}(4) = -2.33(4)^3 + 46.25(4)^2 - 285.22(4) + 1107.80$ $= 557.80$	$(4, 557.8)$
$t \approx 4.89$	$\bar{R}(4.89) = -2.33(4.89)^3 + 46.25(4.89)^2 - 285.22(4.89) + 1107.80$ ≈ 546.56	$(4.89, 546.56)$
$t \approx 8.34$	$\bar{R}(8.34) = -2.33(8.34)^3 + 46.25(8.34)^2 - 285.22(8.34) + 1107.80$ ≈ 594.39	$(8.34, 594.39)$
$t = 9$	$\bar{R}(9) = -2.33(9)^3 + 46.25(9)^2 - 285.22(9) + 1107.80$ $= 588.5$	$(9, 588.5)$

The average annual service revenue per customer is highest at $t \approx 8.34$ or in the year 2008.

b. When the average annual service revenue per customer was highest, how much was each customer paying each month?

Solution The average annual service revenue per customer was highest at $t \approx 8.34$ at a value of 594.39 dollars per customer. However, this is the annual service revenue per customer. To get the monthly service revenue per customer at this time, divide this value by 12 to give

$$\begin{aligned}\frac{\overline{R}(8.34)}{12} &\approx \frac{594.39}{12}\\ &\approx 49.53\end{aligned}$$

or 49.53 dollars per month for each customer.