12.4 Business Applications of Extrema

Question 1: How do you find the optimal dimensions of a product?

Question 2: How do we make decisions about inventory?

When we optimize business functions such as profit, revenue, or average cost, the functions are ultimately derived from financial data produced by the business. Ideally, these data would uniquely determine a function which could be optimized. However, conditions at a business normally do not allow the trends formed by cost and revenue data to be fit perfectly by a function. Costs vary due to fluctuations in the cost of materials, electricity, and labor. Prices may also change leading to fluctuations in revenue and profit. Functions like those in section 12.3 are typically obtained through linear and nonlinear regression.

In this section, the objective functions we wish to minimize describe a particular variable cost and are constructed by examining the application. We'll also look at designing an object with maximum volume subject to limitations on its dimensions. In each case, information about the application will be used to model the objective function. Once the objective function is in place, we'll analyze the function to find its relative extrema using the derivative of the objective function.

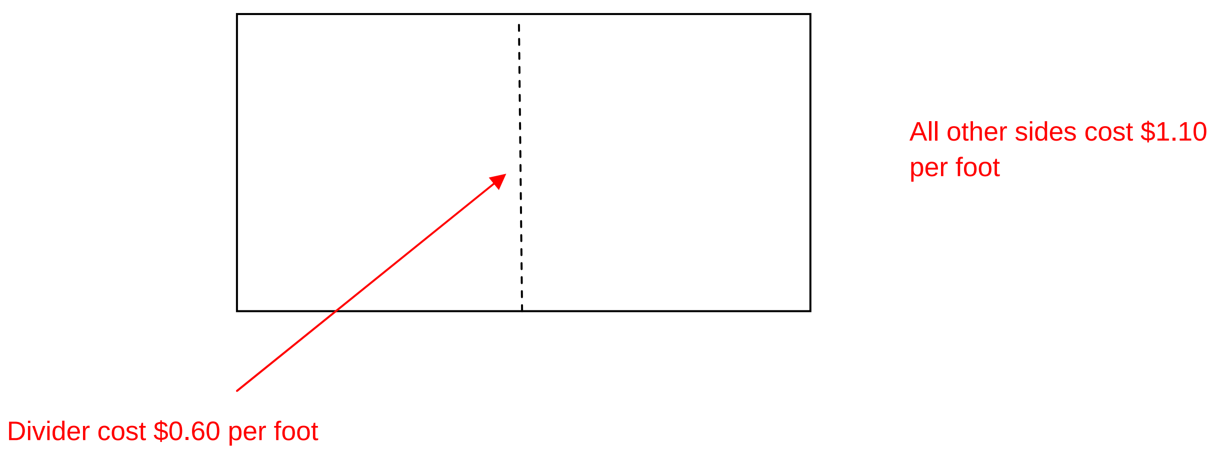
Question 1: How do you find the optimal dimensions of a product?

The size and shape of a product influences its functionality as well as the cost to construct the product. If the dimensions of a product are designed to minimize the cost of materials used to construct the product, then the objective function models the cost of the material in terms of the dimensions of the product.

In 0, we find an objective function for the cost of materials to build a rectangular enclose. By finding the relative minimum of this objective function, we are able to find the dimensions of the enclosure that costs the least amount.

Example 1 Minimize Cost of Materials

A farmer is fencing a rectangular area for two equally sized pens. These pens share a divider that will be constructed from chicken wire costing \$0.60 per foot. The rest of the pen will be built from fencing costing \$1.10 per foot.



If the pens should enclose a total of 400 square feet, what overall dimensions should the pens have to minimize fencing costs?

Solution To find the minimum fencing costs, we must formulate a function that describes the cost of fencing as a function of some variable. In this problem, we have two variables we could use. The width or length of the enclosed region can be the variable. In this example, we'll choose the variable to be the width of the region.

Informally, the cost of fencing is

$$\text{Cost} = \text{Cost of Width Components} + \text{Cost of Length Components}$$

Each of these terms describes the cost of individual components of the fencing around the enclosed area. The first term matches the parts of the fencing along the top and bottom of the figure costing $\$1.10$ per foot. The second term matches the two parts of fencing along the sides costing $\$1.10$ per foot as well as the divider costing $\$0.60$ per foot. A table is useful to help us recognize the function describing the cost of the fencing. In this table, each column will represent the quantities that will vary, the two terms in the cost of fencing as well as the total cost of fencing.

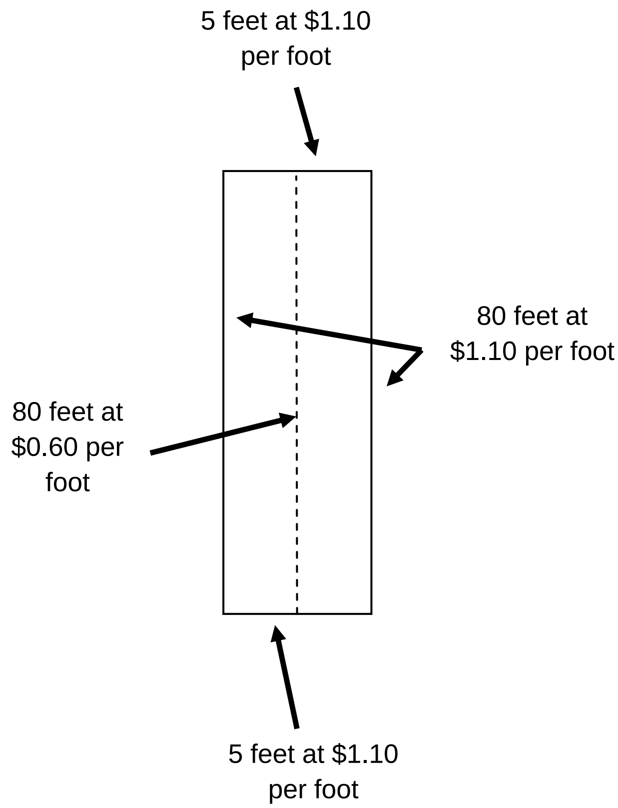
The table below contains five columns for these quantities and five blank rows in which we'll place some values.

Width (feet)	Length (feet)	Cost of Width Components	Cost of Length Components	Cost of Fencing

Start by entering several combinations for length and width that yield 400 square feet. For example, a pen that is 5 feet by 80 feet encloses 400 square feet. Another possibility is a pen that is 10 feet by 40 feet. In general, if the width is w then the length is $\frac{400}{w}$.

Width (feet)	Length (feet)	Cost of Width Components	Cost of Length Components	Cost of Fencing
5	80			
10	40			
20	20			
40	10			
w	$\frac{400}{w}$			

Now let’s calculate the costs involved in fencing a pen that is 5 feet by 80 feet.



The width components will cost

$$\underbrace{1.10}_{\substack{\text{cost per}\\ \text{foot}}} \cdot \underbrace{2 \cdot 5}_{\substack{\text{feet of}\\ \text{fencing}}}$$

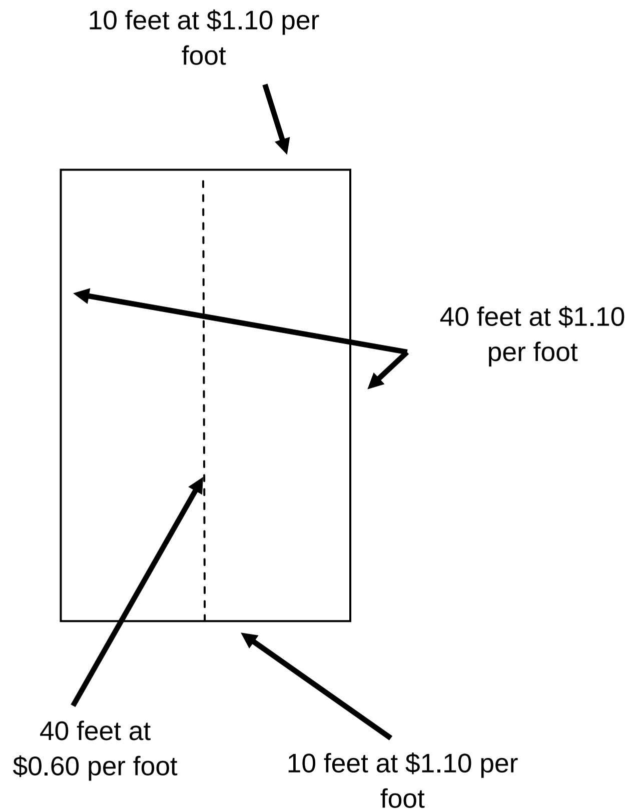
and the length components will cost

$$\underbrace{1.10}_{\substack{\text{cost per}\\ \text{foot}}} \cdot \underbrace{2 \cdot 80}_{\substack{\text{feet of}\\ \text{fence}}} + \underbrace{0.60}_{\substack{\text{cost per}\\ \text{foot}}} \cdot \underbrace{80}_{\substack{\text{feet of}\\ \text{fence}}}$$

to give a total cost of

$$\text{Cost} = 1.10 \cdot 2 \cdot 5 + 1.10 \cdot 2 \cdot 80 + 0.60 \cdot 80$$

The cost involved in fencing a pen that is 10 feet by 40 feet is calculated in a similar manner.



The width components will cost

$$\underbrace{1.10}_{\substack{\text{cost per}\\ \text{foot}}} \cdot \underbrace{2 \cdot 10}_{\substack{\text{feet of}\\ \text{fencing}}}$$

and the length components will cost

$$\underbrace{1.10}_{\substack{\text{cost per}\\ \text{foot}}} \cdot \underbrace{2 \cdot 40}_{\substack{\text{feet of}\\ \text{fence}}} + \underbrace{0.60}_{\substack{\text{cost per}\\ \text{foot}}} \cdot \underbrace{40}_{\substack{\text{feet of}\\ \text{fence}}}$$

to give a cost of

$$\text{Cost} = 1.10 \cdot 2 \cdot 10 + 1.10 \cdot 2 \cdot 40 + 0.60 \cdot 40$$

This expression is almost identical to the earlier expression except amounts of fencing are different.

We can continue to calculate other sized pens to fill out a table of values.

Width (feet)	Length (feet)	Cost of Width Components	Cost of Length Components	Cost of Fencing
5	80	$1.10\cdot 2\cdot 5$	$1.10\cdot 2\cdot 80+0.60\cdot 80$	235
10	40	$1.10\cdot 2\cdot 10$	$1.10\cdot 2\cdot 40+0.60\cdot 40$	134
20	20	$1.10\cdot 2\cdot 20$	$1.10\cdot 2\cdot 20+0.60\cdot 20$	100
40	10	$1.10\cdot 2\cdot 40$	$1.10\cdot 2\cdot 10+0.60\cdot 10$	116
w	$\frac{400}{w}$	$1.10\cdot 2\cdot w$	$1.10\cdot 2\cdot\frac{400}{w}+0.60\cdot\frac{400}{w}$	$1.10\cdot 2\cdot w+1.10\cdot 2\cdot\frac{400}{w}+0.60\cdot\frac{400}{w}$

The last row in the table, the width is labeled as w and the corresponding length must be $\frac{400}{w}$ to ensure the area is 400. The other entries in the last row use the width and length to match the pattern in the rows above.

Based on this pattern, we can define the cost function $C(w)$ as a function of the width w,

$$C(w)=1.10\cdot 2\cdot w+1.10\cdot 2\cdot\frac{400}{w}+0.60\cdot\frac{400}{w}$$

We can simplify the function to

$$C(w) = 2.20w + \frac{880}{w} + \frac{240}{w}$$ Carry out the multiplications in each term

$$= 2.20w + \frac{1120}{w}$$ Combine like terms

We'll need to take the derivative of this function to find the relative minimum. It is easiest to take the derivative with the power rule for derivatives if we rewrite the second term with a negative exponent. With this modification, the cost function is

$$C(w) = 2.20w + 1120w^{-1}$$

The derivative of $C(w)$ is

$$C'(w) = 2.20 - 1120w^{-2}$$

To find the critical points, we need to find where the derivative is equal to zero or undefined. Set the derivative equal to zero and solve for w:

$$2.20 - 1120w^{-2} = 0$$

$$2.20 - \frac{1120}{w^2} = 0$$ Rewrite the negative exponent as a positive exponent

$$2.20w^2 - \frac{1120}{w^2} \cdot w^2 = 0 \cdot w^2$$

$$2.20w^2 - 1120 = 0$$ Clear the fraction by multiplying each term by w^2

To solve the quadratic equation $2.20w^2 - 1120 = 0$, we could use the quadratic formula. However, it is easier to isolate w^2:

$2.20w^2 - 1120 = 0$

$2.20w^2 = 1120$ — Add 1120 to both sides

$w^2 = \frac{1120}{2.20}$ — To isolate w^2, divide both sides by 2.20

$w = \pm\sqrt{\frac{1120}{2.20}}$

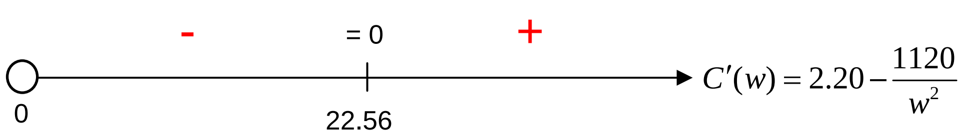
$w \approx \pm 22.56$ — Take the square root of both sides

The negative critical value is not a reasonable dimension for the width of the pen so we can ignore it.

If the derivative is rewritten as $C'(w) = 2.20 - \frac{1120}{w^2}$, we observe that it is undefined at $w = 0$. Like negative values, a width of zero is not a reasonable dimension for a pen. Ignoring this value, we have only one critical value, $w \approx 22.56$, in the domain of this problem.

This critical value may correspond to a relative minimum or a relative maximum. We can use the first derivative test to determine which type of extrema this critical value matches.

If we test the first derivative on either side of the critical value, we can establish where $C(w)$ is increasing and decreasing.



This first derivative test indicates that the function is decreasing and then increasing so $w \approx 22.56$ corresponds to a relative minimum.

The expression for the length is $\frac{400}{w}$. Using the width, $w \approx 22.56$ feet, the length is calculated as $\frac{400}{22.56} \approx 17.73$ feet. These dimensions yield a minimum total cost of

$$\begin{aligned} C(22.56) &\approx 2.20(22.56) + \frac{1120}{22.56} \\ &\approx 99.28 \text{ dollars} \end{aligned}$$

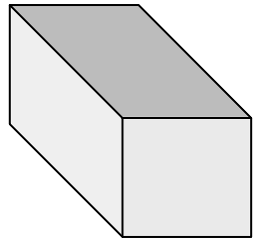
For some businesses, the goal of the design process is not a product that costs as little as possible. Instead, they maximize or minimize some characteristic of the product with respect to the dimensions of the product. For instance, a newspaper publisher might maximize the area of the printed page with respect to the margins and dimensions of the page. By doing this, they maximize the area used to print news as well as the area used for advertisements.

In the next example, we find the dimensions a piece of carry-on baggage that maximizes the volume inside the bag and meets airline requirements for the dimensions.

Example 2 Maximize Volume

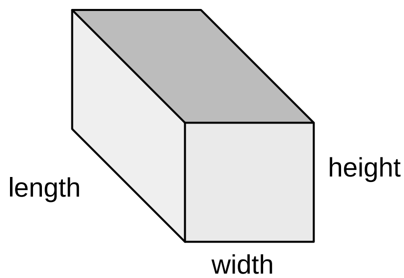
Most airlines charge to check baggage on flights. To avoid these charges, passengers pack as much as possible into their carry-on bags. However, airlines also limit the size of these bags. American Airlines limits the linear dimensions (defined as the sum of the length, width and height) to 45 inches.

A manufacturer wishes to produce a carry-on bag whose linear dimensions are 45 inches. The shape of the bag is a rectangular solid, like the one pictured, below whose ends are squares.



What are the dimensions of the bag if its volume is to be as large as possible?

Solution We want to maximize the volume of the carry-on bag. To help us understand the relationships between the dimensions, let's look at a particular carry-on bag.



Suppose the dimensions of the square ends are 5 inches by 5 inches. Since the sum of the length, width, and height must be 45 inches, we know the length must be $45-5-5=35$ inches.

The volume of this carry-on bag is the product of the length, width, and height so this bag is

$$\text{Volume}=(5\text{ inches})(5\text{ inches})(35\text{ inches})=875\text{ in}^3$$

Let's look at a second possibility. Suppose the dimensions of the square end are 10 inches by 10 inches. The length must be $45-10-10=25$ inches. The volume of this carry-on bag is

$$\text{Volume}=(10\text{ inches})(10\text{ inches})(25\text{ inches})=2500\text{ in}^3$$

Notice that these dimensions result in a larger volume.

We can continue this process to see if the volume continues to increase as the square end gets larger. To keep track of the information, let's enter this information into a table.

Width	Height	Length	Volume
5	5	$45-5-5$ or 35	$(5)(5)(45-5-5)$ or $(5)(5)(35)=875$
10	10	$45-10-10$ or 25	$(10)(10)(45-10-10)$ or $(10)(10)(25)=2500$
15	15	$45-15-15$ or 15	$(15)(15)(45-15-15)$ or $(15)(15)(15)=3375$
20	20	$45-20-20$ or 5	$(20)(20)(45-20-20)$ or $(20)(20)(5)=2000$

As the dimensions of the square end increases, the volume rises and then falls. It appears that the optimal dimensions are around 15 inches by 15 inches by 15 inches.

To find a more exact answer, we need to come up with an expression for the volume. We do this by identifying the variable as the width w and look for pattern in each column of the table.

Width	Height	Length	Volume
5	5	$45-5-5$ or 35	$(5)(5)(45-5-5)$ or $(5)(5)(35)=875$
10	10	$45-10-10$ or 25	$(10)(10)(45-10-10)$ or $(10)(10)(25)=2500$
15	15	$45-15-15$ or 15	$(15)(15)(45-15-15)$ or $(15)(15)(15)=3375$
20	20	$45-20-20$ or 5	$(20)(20)(45-20-20)$ or $(20)(20)(5)=2000$
w	w	$45-w-w$	$(w)(w)(45-w-w)$

If the width corresponds to w, so must the height since the ends are square. If we subtract these dimensions from 45, we get the length $45-w-w$. The volume is the product of these dimensions.

Using this pattern, define the volume V as

$$V(w)=(w)(w)(45-w-w)$$

This simplifies to

$$\begin{aligned} V(w) &= w^2(45-w-w) && \text{Multiply the first two factors} \\ &= w^2(45-2w) && \text{Combine like terms} \\ &= 45w^2-2w^3 && \text{Multiply and remove the parentheses} \end{aligned}$$

It is much easier to take the derivative of this function once it has been simplified. Using the basic rules for derivatives, the derivative of $V(w)=45w^2-2w^3$ is calculated as

$$\begin{aligned} V'(w) &= 45(2w)-2(3w^2) \\ &= 90w-6w^2 \end{aligned}$$

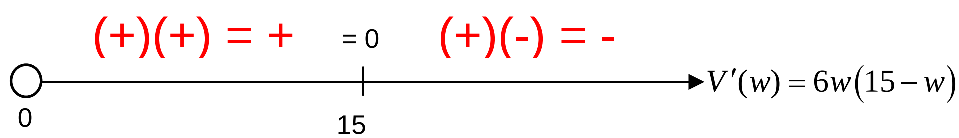
This derivative, $V'(w) = 90w - 6w^2$, is set equal to zero to find the critical values for the function. Since the derivative is a polynomial, there are no values of w for which the derivative is undefined. Additionally, all critical values must be positive since the width must be a positive number.

Set the derivative equal to zero and solve for w:

$90w - 6w^2 = 0$	Set the derivative equal to zero
$6w(15 - w) = 0$	Factor the greatest common factor, $6w$
$6w = 0 \quad 15 - w = 0$	Set each factor equal to zero and solve for w
$w = 0 \qquad 15 = w$	

Only the critical value at $w = 15$ is a reasonable dimension of the carry-on.

Now let's use the first derivative test to determine if the critical value is a relative minimum or a relative maximum:



The volume function increases initially and then decreases after $w = 15$. This matches the behavior we saw in the last column of the table. It also tells us that the critical value corresponds to a relative maximum. The optimal solution occurs when the square end is 15 inches by 15 inches and the length is $45 - 15 - 15$ or 15 inches. The volume at these dimensions is $(15 \text{ inches})(15 \text{ inches})(15 \text{ inches})$ or 3375 in^3.

Question 2: How do we make decisions about inventory?

Most businesses keep a stock of goods on hand, called inventory, which they intend to sell or use to produce other goods. Companies with a predictable demand for a good throughout the year are able to meet the demand by having an adequate supply of the good. A large inventory costs money in storage cost and carrying low inventory subjects a business to undesirable shortages called stockouts. A company's inventory level seeks to balance the storage cost with costs due to shortages.

A light emitting diode (LED) is a light source that is used in many lighting applications. In particular, LEDs are used to produce very bright flashlights used by law enforcement, fire rescue squads and sports enthusiasts. Suppose a manufacturer of LED flashlights needs 100,000 LED bulbs annually for their flashlights.

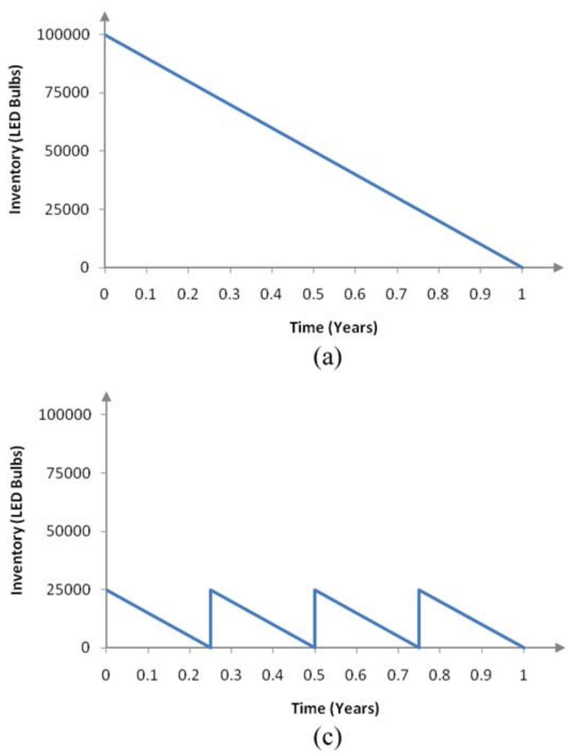


Figure 1 – Inventory levels at the manufacturer over a one year period. In (a), all bulbs are made at the beginning of the year and used at a constant rate throughout the year. In (b), two batches of 50,000 are made. In (c), 4 batches of 25,000 are made. In (d), 8 batches of 12,500 are made. In each case, when the inventory reaches 0, the next batch is manufactured instantaneously.

The company may manufacture all of the LED bulbs at one time or in smaller batches throughout the year to meet its annual demand. If the manufacturer makes all of the bulbs at one time they will have a supply to make flashlights throughout the year, but will need to store bulbs for use later in the year. On the other hand, if they make bulbs at different times throughout the year, the factory will need to be retooled to make each batch.

Since the manufacturer does not need to make LED bulbs continuously throughout the year, they will use the manufacturing capacity for other purposes during the rest of the year. When they do manufacture the bulbs, the factory will need to be set up for this purpose. For this manufacturer, it costs $\$5000$ to set up the factory to manufacture LED bulbs. This amount covers all costs involved in setting up the factory such as retooling costs, diminished capacity while the production line is retooled, and labor costs. Each time the company manufactures a batch of bulbs, they incur this cost. Obviously, the more times the plant retools, the higher the total cost is.

The company pays a holding cost to store any LED bulbs not used. The holding cost includes the taxes, insurance, and storage costs that change as the number of units stored changes. If they manufacture all of the bulbs at one time, they will have to store more bulbs throughout the year. For this manufacturing plant, it costs $\$1$ to hold a bulb for one year.

We'll assume that the company's production cost does not change throughout the year. In other words, whether they manufacture the bulbs all at once or in several batches throughout the year, it will cost them the same amount per unit to manufacture the bulbs. For this plant, it costs $\$25$ to manufacture an LED bulb.

The total cost for the manufacturer is the sum of the set up, holding and production costs. For other manufacturers, other costs may be included in this sum. However, we'll assume that the total cost for this plant is restricted to these three costs. If they produce more LED bulbs in each batch, the set up costs will be lower but they will pay a higher holding cost. On the other hand, if they produce fewer bulbs in each batch they will lower the holding cost. This decrease in holding cost is accompanied by an increase in the set up cost since more batches will be needed to meet the annual demand. The manufacturer faces a simple question: how many units should they manufacture in each batch so that their total cost is minimized? The batch size that results in the lowest total cost is called the economic lot size.

Example 3 Find the Economic Lot Size

A manufacturing plant needs to make 100,000 LED bulbs annually. Each bulb costs $25 to make and it costs $5000 to set up the factory to produce the bulbs. It costs the plant $1 to store a bulb for 1 year. How many bulbs should the plant produce in each batch to minimize their total costs?

Solution To find the economic lot size, we need to analyze the costs for the plant and model these costs.

What kind of costs will be incurred? Three different types of costs are described in this example. A set up cost of $5000 is incurred to set up the factory. A production cost of $25 per bulb is incurred for labor, materials, and transportation. Since the demand for bulbs occurs throughout the year, we'll need to store some of them in a warehouse at a holding cost of $1 per bulb for a year.

Let's try to solve this naively and simply produce all 100,000 bulbs in one batch.

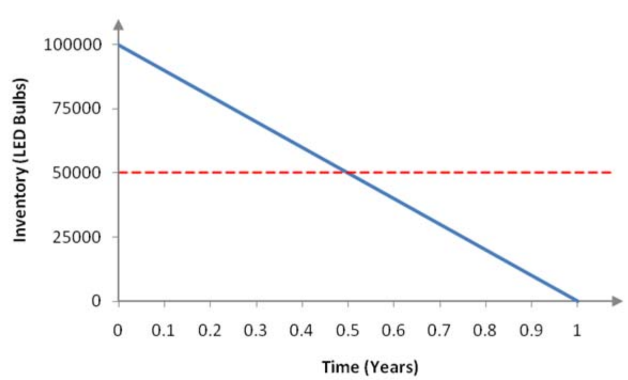


Figure 2 - If all of the LED bulbs are produced in one batch, the average inventory is 50,000 bulbs.

If the demand throughout the year is uniform, we can expect to have an average inventory of

$$\text{average inventory} = \frac{100,000 + 0}{2} = 50,000$$

It will cost \$1 to store each of these bulbs for holding costs of 50,000 × \$1 or \$50,000.

If all of the bulb are produced in one batch, the production line will need to be set up once for a cost of \$5000.To produce 100,000 bulbs at a cost of \$25 per bulb will cost 100,000 × \$25 or \$2,500,000. The total cost to produce one batch of 100,000 bulbs is the sum of these costs,

$$\text{Total Cost} = \underbrace{5000}_{\text{set up}} + \underbrace{100,000(25)}_{\text{production}} + \underbrace{50,000(1)}_{\text{holding}} = 2,555,000$$

The largest cost in this sum is the production cost. If the cost to make a bulb is fixed and we don't change the number of bulbs produced each year, changing the batch size won't affect this term.

We can lower the holding cost by producing fewer bulbs in each batch. But this increases the set up cost since the production line will need to be set up more often.

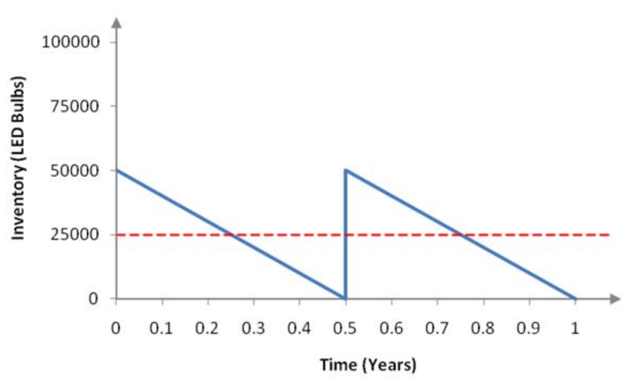


Figure 3 -If the LED bulbs are produced in two batches, the average inventory is reduced to 25,000 bulbs.

If the batch size is reduced to 50,000, the average inventory is reduced to

$$\text{average inventory} = \frac{50,000+0}{2} = 25,000$$

for storage costs of $25,000 \times \$1$ or \$25,000. The production line will need to be set up twice at a cost of $2 \times \$5000$ or \$10,000.

We will still produce 100,000 bulbs at a cost of \$25 per bulb. This will cost $100,000 \times \$25$ or \$2,500,000. The total cost is now

$$\text{Total Cost} = \underbrace{2(5000)}_{\text{set up}} + \underbrace{100,000(25)}_{\text{production}} + \underbrace{25,000(1)}_{\text{holding}} = 2,535,000$$

In this case, the holding cost dropped by \$25,000, but the set up cost increased by \$500. This results in a lower total cost.

Continuing with this strategy, we can fill out the following table:

Size of Batch	Set Up Cost	Production Cost	Holding Cost	Total Cost
100,000	5000	100,000(25)	50,000(1)	$5000+100,000(25)+50,000(1)=2,555,000$
50,000	2(5000)	100,000(25)	25,000(1)	$2(5000)+100,000(25)+25,000(1)=2,535,000$
10,000	10(5000)	100,000(25)	5000(1)	$10(5000)+100,000(25)+5000(1)=2,555,000$
1000	100(5000)	100,000(25)	500(1)	$100(5000)+100,000(25)+500(1)=3,000,500$

As the lot size decreases, the set up cost increases and the holding cost decreases. Initially, the holding cost is much higher. But for smaller batch sizes, the set up cost is much higher. Somewhere in the middle is a batch size whose total cost is as small as possible.

To find the batch size Q that minimizes the total cost, we need to find a function that models the total cost as a function of Q. Examine the patterns in the table. Each set up cost in the second column is a product of \$$5000$ and the number of batches that will be produced. We get the number of batches by dividing $100{,}000$ by the batch size, $\frac{100{,}000}{Q}$. This means a batch size of Q will have a set up cost of

$$\text{Set Up Cost} = \frac{100{,}000}{Q}(500)$$

In every row of the second column, the production cost is the same. So changing the size of the batch has no effect on the production cost.

The holding cost is the product of the average inventory and the unit holding cost. If the batch size is Q, the holding cost is

$$\text{Holding Cost} = \frac{Q}{2}(1)$$

Let's add these expressions to the table.

Size of Batch	Set Up Cost	Production Cost	Holding Cost	Total Cost
100,000	5000	100,000(25)	50,000(1)	$5000+100,000(25)+50,000(1)=2,555,000$
50,000	2(5000)	100,000(25)	25,000(1)	$2(5000)+100,000(25)+25,000(1)=2,535,000$
10,000	10(5000)	100,000(25)	5000(1)	$10(5000)+100,000(25)+5000(1)=2,555,000$
1000	100(5000)	100,000(25)	500(1)	$100(5000)+100,000(25)+500(1)=3,000,500$
Q	$\frac{100,000}{Q}(5000)$	100,000(25)	$\frac{Q}{2}(1)$	$\frac{100,000}{Q}(5000)+100,000(25)+\frac{Q}{2}(1)$

Notice that the expression in the last row for the total cost preserves the pattern for all values of Q above it.

The total cost as a function of the batch size Q is

$$TC(Q)=\frac{100,000}{Q}(5000)+100,000(25)+\frac{Q}{2}(1)$$

Before we take the derivative to find the critical points, let's simplify this function,

$$\begin{aligned}TC(Q)&=\frac{500,000,000}{Q}+2,500,000+\frac{Q}{2}\\&=500,000,000Q^{-1}+2,500,000+\frac{1}{2}Q\end{aligned}$$

Rewrite with negative exponents and change the division in the last term to multiplication

Using the Power Rule, the derivative is computed as

$$TC'(Q) = -500,000,000Q^{-2} + \frac{1}{2}$$ Use the Power Rule as well as the Product with a Constant Rule

$$= -\frac{500,000,000}{Q^2} + \frac{1}{2}$$ Switch the negative exponent back to a positive exponent

Notice that the production cost drops out of the critical point calculation meaning the production cost has nothing to do with the economic lot size. This function is undefined at $Q = 0$. But a batch size of 0 is not a reasonable answer since a total of $100{,}000$ bulbs must be made.

More critical points can be found by setting the derivative equal to 0 and solving for Q:

$$-\frac{500,000,000}{Q^2} + \frac{1}{2} = 0$$ Set derivative equal to zero

$$-1,000,000,000 + Q^2 = 0$$ Multiply both sides by $2Q^2$ to clear fractions

$$Q^2 = 1,000,000,000$$ Add 1,000,000,000 to both sides

$$Q \approx 31,623$$ Square root both sides of equation

Only the quantity $31{,}623$ bulbs makes sense. But is this critical point a relative minimum or a relative maximum?

To check, we'll substitute the critical point into the second derivative and determine the concavity at the critical point. Starting from the first derivative, $TC'(Q) = -500,000,000Q^{-2} + \frac{1}{2}$, the second derivative is

$$TC''(Q) = 1,000,000,000Q^{-3}$$

$$= \frac{100,000,000}{Q^3}$$

At the critical point,

$$TC''(31,623) = \frac{1,000,000,000}{31,623^3} > 0$$

The second derivative is positive since the numerator and denominator are both positive. A function that has a positive second derivative at a critical point is concave up. This tells us that $Q = 31,623$ is a relative minimum.

The total cost at that point is

$$\begin{aligned} TC(31,623) &\approx \frac{500,000,000}{31,623} + 2,500,000 + \frac{31,623}{2} \\ &\approx 2,531,622.78 \end{aligned}$$

A lot size of approximately 31,623 bulbs leads to the lowest total cost possible of $2,531,622.78. ■

Some companies purchase their inventory from manufacturers. Instead of deciding how many units to manufacture in each batch, they must decide how many units they should order and when they should order the units. For a business like this, the set up cost is replaced by the ordering cost.

The ordering cost includes all costs associated with ordering inventory such as developing and processing the order, inspecting incoming orders, and paying the bill for the order. Larger companies may also have purchasing departments. The ordering cost also includes the cost of the personnel and supplies for the purchasing department.

Ordering more often lowers holding cost since it results in lower inventory levels. This also raises ordering cost since more orders must be placed. The total cost is minimized at an order size that balances the ordering cost and the holding cost. This order size is called the economic order quantity.

Example 4 Find the Economic Order Quantity

The annual demand for a particular wine at a wine shop is 900 bottles of wine. It costs \$1 to store one bottle of wine for one year. It costs \$5 to place an order for a bottle of wine. A bottle of wine costs an average of \$15. How many bottles of wine should be ordered in each order to satisfy demand and to minimize cost?

Solution Upon examining this problem, it might appear that the wine shop should simply order 900 bottles of the wine once a year. While this is a possible solution and lowers the ordering cost, it would incur a large holding cost. The wine shop could also place 18 orders of 50 bottles each. This would lower the holding cost, but increase the ordering cost. Another possibility would be to order each bottle individually. This would lower the holding cost even more, but increase the ordering cost. The appropriate order size will balance the holding cost and the ordering cost so that the total cost is as small as possible.

To solve this problem, we need to find a total cost function. Once we have this cost function, we'll take the derivative of the function to find the critical points and locate the relative minimum. For this problem, we need to vary the order size to see the effect on the costs. For this reason, the variable in this problem will be the order size Q. Let's calculate the costs for several different values of Q to see the relationship between the order size and the total cost.

Suppose we make a single order of 900 bottles of wine. Since we are making a single order, the ordering costs will be \$5. However, it will take the entire year for all of these bottles to be sold. The average amount on hand will be

$$\text{average inventory} = \frac{900+0}{2} = 450$$

Thus the storage costs will be 450 bottles $\times$ \$$1$ per bottle to store. Each bottle of wine costs an average of \$$15$, so 900 bottles will cost $900(\$15)$ or \$$13{,}500$.

The total cost is

$$\text{Total Cost} = 5 + 450(1) + 13500$$

If we make two orders of 450 bottles each, the reordering cost will be $2 \times$ \$$5$ and the storage costs will be 225 bottles $\times$ \$$1$ per bottle. Assuming the cost of the wine does not change throughout the year, the wine will still cost \$$13{,}500$. For this order size, the total ordering cost is

$$\text{Total Cost} = 2(5) + 225(1) + 13500$$

Using this strategy, we can fill out the table below:

Order Size	Ordering Cost	Storage Cost	Cost of Wine	Total Cost
900	5	450(1)	13,500	$5 + 450(1) + 13500 = 13955$
450	2(5)	225(1)	13,500	$2(5) + 225(1) + 13500 = 13735$
225	4(5)	112.5(1)	13,500	$4(5) + 112.5(1) + 13500 = 13632.5$
50	18(5)	25(1)	13,500	$18(5) + 25(1) + 13500 = 13615$
10	90(5)	5(1)	13500	$90(5) + 5(1) + 13500 = 13955$
1	900(5)	0.5(1)	13500	$900(5) + 0.5(1) + 13500 = 18000.50$

Look at each line of this table. As the order size decreases, the reordering costs increase and the storage costs decrease. The sum of all costs starts at \$13955 and decreases initially as the storage costs drop. However the reordering costs begin to build up eventually causing the total ordering cost to increase to \$4500.50 for an order size of 1 bottle.

Based on the table, we can guess that an order around 50 bottles might lead to the lowest total ordering cost. To make this more exact, we need to find the total cost as a function of the order quantity Q. The ordering cost come from the number of orders times the cost per order. We can find the number of orders by dividing 900 by the order size or $\frac{900}{Q}$. The ordering cost is

$$\text{Ordering Cost} = \frac{900}{Q}(5)$$

The holding cost are simply the average inventory times the cost per bottle to store or

$$\text{Holding Cost} = \frac{Q}{2}(1)$$

The total cost function is the sum of these costs and the cost of the wine,

$$TC(Q) = \frac{900}{Q}(5) + \frac{Q}{2}(1) + 13500$$

Let’s add these quantities to our table to see how they match up with the numbers we found before:

Order Size	Reordering Costs	Storage Costs	Cost of Wine	Total Cost
900	5	450(1)	13,500	$5+450(1)+13500=13955$
450	2(5)	225(1)	13,500	$2(5)+225(1)+13500=13735$
225	4(5)	112.5(1)	13,500	$4(5)+112.5(1)+13500=13632.5$
50	18(5)	25(1)	13,500	$18(5)+25(1)+13500=13615$
10	90(5)	5(1)	13500	$90(5)+5(1)+13500=13955$
1	900(5)	0.5(1)	13500	$900(5)+0.5(1)+13500=18000.50$
Q	$\frac{900}{Q}(5)$	$\frac{Q}{2}(1)$	13500	$\frac{900}{Q}(5)+\frac{Q}{2}(1)+13500$

Each ordering cost in the second column consists of the \$5 cost to order times a value. This value is the number of orders you will need to make during the year. Each holding cost in the third column consists of the \$1 cost to store one bottle for one year, times the average inventory. The expression we have written for the total costs preserves these patterns.

Before we find the critical points of $TC(Q)$, let's simplify the function to make the derivative easier to find. Carry out the multiplication in the first term and use negative exponents in the first term to give

$$TC(Q)=4500Q^{-1}+\tfrac{1}{2}Q+13500$$

The derivative of this function is

$$TC'(Q) = -4500Q^{-2} + \tfrac{1}{2}$$

Use the Power Rule for Derivatives

$$= -\frac{4500}{Q^2} + \frac{1}{2}$$

Change the negative exponent to a positive exponent

To find the critical points, we need to find where the derivative is undefined or equal to 0. This function is undefined at $Q = 0$, but an order size of 0 is not a reasonable order since we need 900 bottles annually. Setting the derivative equal to 0 and solving for Q yields

$$-\frac{4500}{Q^2} + \frac{1}{2} = 0$$

Multiplying both side by $2Q^2$ to clear fractions

$$-9000 + Q^2 = 0$$

Add 9000 to both sides

$$Q^2 = 9000$$

$$Q = \pm\sqrt{9000}$$

Square root both sides

This is approximately ± 94.87. Clearly, a negative order size makes no sense. The only reasonable critical point for this function is $Q \approx 94.87$.

To insure that this is a relative minimum and not a relative maximum, we'll apply the second derivative test. The second derivative is

$$TC''(Q) = 9000Q^{-3}$$

$$= \frac{9000}{Q^3}$$

Change the negative exponent to a positive exponent

At the critical point, the second derivative is

$$TC''(94.87) = \frac{9000}{94.87^3} > 0$$

so the original function $TC(Q)$ is concave up. The critical point at $Q \approx 94.87$ is a relative minimum.

Should the wine shop order 94.87 bottles of wine? Bottles of wine are sold in integer amounts so we have two options: order 94 bottles (and not meet the demand) or order 95 bottles (and have a few bottles extra at the end of the year). You might choose to order 95 per order to simply insure that you satisfy all customers, but which option is cheapest?

$$TC(94) = \frac{4500}{94} + \tfrac{1}{2}\left(94\right) + 13500 \approx \$13594.872$$

$$TC(95) = \frac{4500}{95} + \tfrac{1}{2}\left(95\right) + 13500 \approx \$13594.868$$

In deciding which option is better, you'll need to balance what is the biggest benefit and what are the costs of the decision. In this case, an order size of 95 is slightly cheaper than an order size of 94 and ensures that all customers are satisfied.