4.1 Solving a System of Linear Inequalities

Question 1: How do you graph a linear inequality?

Question 2: How do you graph a system of linear inequalities?

In Chapter 2, we were concerned with systems of linear equations. In this section, we'll change the equal signs in systems of linear equations to inequalities to yield systems of linear inequalities. Systems of linear inequalities occur frequently in business where production is constrained by the availability of materials, labor, or transportation.

For instance, over the last twenty years craft beer has become very popular. One of the largest craft breweries produces five different beers throughout the year. The monthly capacity of the brewery is 50,000 barrels of beer (one barrel is 31 gallons). If \( x_1 \) through \( x_5 \) represent the monthly production of each of the different types of beer in barrels, we can write

\[
x_1 + x_2 + x_3 + x_4 + x_5 = 50,000
\]

The sum of the variables represents the total monthly production, so setting this sum equal to 50,000 tells us the brewery is operating at its monthly capacity of 50,000 barrels.

However, the brewery may not operate at its full capacity. If the brewery is producing less than 50,000 barrels of beer, we would write \( x_1 + x_2 + x_3 + x_4 + x_5 < 50,000 \).

The inequality points to the smaller quantity so the expression on the left, the total monthly production, is smaller than 50,000 barrels. If the brewery is operating above capacity, the inequality
The total monthly production is greater than 50,000 barrels. Inequalities with < or > are called strict inequalities.

Inequalities may also involve a combination of <, >, and =. By writing

\[ x_1 + x_2 + x_3 + x_4 + x_5 \leq 50,000 \]

we know that the brewery is operating at or below capacity. We would read this as the total monthly production, \( x_1 + x_2 + x_3 + x_4 + x_5 \), is less than or equal to 50,000. If the inequality is reversed, \[ x_1 + x_2 + x_3 + x_4 + x_5 \geq 50,000, \]

we say that the total monthly production is greater than or equal to 50,000 barrels.

In this section, we'll examine solutions of inequalities involving two variables. When only two variables are involved, we can graph the solutions on a rectangular coordinate system and shade the region of the graph where the ordered pairs satisfy the inequality. The ordered pairs satisfying the inequality result in a true statement when they are substituted into the inequality.

Let's check to see if the ordered pair \((3,2)\) satisfies the inequality \(x - 2y < 5\). By setting \(x = 3\) and \(y = 2\) in the inequality, we get the statement

\[ 3 - 2(2) < 5 \]

The left side of the inequality is equal to -1. This value is less than 5 so the inequality is true and the ordered pair satisfies the inequality. The ordered pair \((6,-2)\) does not satisfy the inequality \(x - 2y < 5\) since \[ 6 - 2(-2) < 5 \]
is not a true statement.

In a graph of the inequality $x - 2y < 5$, all ordered pairs satisfying the inequality lie in the shaded region in Figure 2. Ordered pairs that do not satisfy the inequality lie in the unshaded region or on the dashed border of the shaded region.

Figure 2 - All ordered in the shaded region satisfy the inequality $x - 2y < 5$. Ordered pairs on the dashed line like $(5, 0)$ do not satisfy the inequality $x - 2y < 5$. 
Question 1: How do you graph a linear inequality?

To graph the solution to an inequality on a rectangular coordinate system, the inequality must contain only two variables. The craft brewery inequalities discussed above contained five variables since the brewery produced five different types of beer. Let’s simplify the inequalities to only two variables so that we can view the solutions to the inequality on a rectangular coordinate system. We’ll do this by modifying the number of beers a particular brewery produces.

Suppose a craft brewery has a monthly capacity of 50,000 barrels of beer and produces two styles of beer, a pale ale and a porter. If the brewery is not to exceed the monthly capacity, we know that

\[ x_1 + x_2 \leq 50,000 \]

where \( x_1 \) is the number of barrels of pale ale produced each month, and \( x_2 \) is the number of barrels of porter produced each month.

Using \( x_1 \) and \( x_2 \) may seem confusing, but we could have easily written this same inequality as

\[ x + y \leq 50,000 \]

where \( x \) is the number of barrels of pale ale produced each month, and \( y \) is the number of barrels of porter produced each month. The names of the variables are irrelevant. However, since we want to be able to generalize two variable problems to problems with more than two variables, we’ll stick with subscripts for this example.

When an inequality in two variables is graphed, we start by changing the inequality to an equation. The equation for this craft brewery is

\[ x_1 + x_2 = 50,000 \]
We’ll graph this equation with $x_1$ on the horizontal axis (the independent variable) and $x_2$ on the vertical axis (the dependent variable). It is perfectly acceptable to switch the variables on the axes. The key is to pick an axis for each variable and to stick with it throughout the entire problem.

The easiest way to graph this equation is by using the intercepts. By setting each variable equal to 0, we can find the corresponding value for the other variable:

\[
\begin{align*}
\text{Set } x_2 &= 0 \\
\text{Then} \\
0 + 0 &= 50,000 \\
x_1 &= 50,000 \\
(x_1, x_2) &= (50,000, 0)
\end{align*}
\]

\[
\begin{align*}
\text{Set } x_1 &= 0 \\
\text{Then} \\
0 + x_2 &= 50,000 \\
x_2 &= 50,000 \\
(x_1, x_2) &= (0, 50,000)
\end{align*}
\]

Drawing a line through these two points gives us the following graph:

![Graph of $x_1 + x_2 = 50,000$](image)

Figure 3 - The line $x_1 + x_2 = 50,000$ passes through the intercepts at $(0, 50,000)$ and $(50,000, 0)$. 
For every point on this line, the sum of $x_1$ and $x_2$ is 50,000. For instance, the point $(25000, 25000)$ is on this line since $25,000 + 25,000 = 50,000$. Points that are not on the line have a sum that is either greater than or less than 50,000. The line in the graph forms the border between points where the sum is greater than 50,000 and points where the sum is less than 50,000.

To determine which side is which, pick a convenient point on the graph that is not on the line. Test this point in the original inequality. If the inequality is true, then every point on the side of the line the test point comes from is in solution to the inequality. If the inequality is false when the test point is substituted, all points on the opposite side of the line from the test point are in the solution for the inequality. By shading the half plane where the test point is true, we can indicate all of the ordered pairs in the solution set of the inequality.

When the inequality includes an equal sign (like $\leq$ or $\geq$) we draw the line separating the two half planes with a solid line. This tells us that the line corresponding to the border, $x_1 + x_2 = 50,000$, is included in the solution set of the inequality.

For the inequality $x_1 + x_2 \leq 50,000$, a convenient test point is $(0, 0)$. Set $x_1 = 0$ and $x_2 = 0$ to yield $0 + 0 \leq 50,000$. Since this is a true statement, all of the points on this side of the line satisfy the inequality. To show this graphically, we shade all of the points on that side in the graph and graph the border as a solid line.
If the inequality is a strict inequality (with no equal sign), the line separating the two half planes is drawn with a dashed line. This tells you that the border is not included in the solution set. For instance, the solution set to the inequality $x_1 + x_2 < 50,000$ is shown in Figure 5.
Graphing a Linear Inequality in Two Variables

1. Identify the independent and dependent variables. Begin the graph of the solution set by labeling the independent variable on the horizontal axis and the dependent variable on the vertical axis.

2. Change the inequality to an equation by replacing the inequality with an equal sign.

3. Graph the equation using the intercepts or another convenient method. If the inequality is a strict inequality, like < or >, graph the line with a dashed line. If the inequality includes an equal sign, like ≤ or ≥, graph the line as a solid line.

4. Pick a test point to substitute into the inequality. Test points that include zeros are easiest to work with. This test point must not be a point on the line.

5. If substituting the test point into the inequality makes it true, shade the side of the line containing the test point. If substituting the test point into the inequality makes it false, shade the side of the line that does not contain the test point.
Example 1  Graph the Linear Inequality

Graph $15x + 20y > 300$.

Solution This linear inequality uses two variables, $x$ and $y$. We'll choose $x$ as the independent variable and $y$ as the dependent variable. This means that the horizontal axis will be labeled with $x$, and the vertical axis labeled with $y$.

To graph the border between the two half planes, we need to graph $15x + 20y = 300$. This is done by solving for $y$:

$$15x + 20y = 300$$

$$20y = -15x + 300$$  Subtract $15x$ from both sides

$$y = -\frac{3}{4}x + 15$$  Divide each term by $20$ and simplify the fraction to $-\frac{3}{4}$

In slope-intercept form we recognize the slope, $-\frac{3}{4}$, and vertical intercept, 15, and use them to graph the line. Since the inequality is a strict inequality, the line must be graphed as a dashed line.

For some equations it may be easier to find the intercepts of the equation:

<table>
<thead>
<tr>
<th>$x$ - Intercept</th>
<th>$y$ - Intercept</th>
</tr>
</thead>
<tbody>
<tr>
<td>Set $y = 0$</td>
<td>Set $x = 0$</td>
</tr>
<tr>
<td>Then $15x + 20(0) = 300$</td>
<td>Then $15(0) + 20y = 300$</td>
</tr>
<tr>
<td>$15x = 300$</td>
<td>$20y = 300$</td>
</tr>
<tr>
<td>$x = 20$</td>
<td>$y = 15$</td>
</tr>
<tr>
<td>$(x, y) = (20, 0)$</td>
<td>$(x, y) = (0, 15)$</td>
</tr>
</tbody>
</table>
Using the slope-intercept method or the intercept method, the graph of $15x + 20y = 300$ looks the same.

![Graph of the line $15x + 20y = 300$.](image)

Figure 6 - The line $15x + 20y = 300$ is the border between half planes. Note the slope of the line and the intercepts. Either method gives the same line. Since the inequality is a strict inequality, the line will be drawn as a dashed line.

Any point that is not on the line can be the test point. We’ll test the point $(5, 5)$. Putting this ordered pair into the inequality yields

$$15(5) + 20(5) > 300$$

$$175 > 300 \quad \text{False}$$

Since this ordered pair does not satisfy the inequality, shade the portion of the plane on the other side of the line from the test point.
Example 2  Find and Graph the Linear Inequality

At a craft brewery, four different ingredients are combined to create beer. Yeast, malted grain, hops and water are mixed, cooked, and fermented in large kettles. The production of beer is limited by several factors. First, the size of the equipment limits the number of barrels that can be produced to 50,000 barrels per month. If $x_1$ barrels of pale ale are produced and $x_2$ barrels of porter are produced, we know that

$$x_1 + x_2 \leq 50,000.$$  

Production is also limited by the availability of grain, the capacity to ship grain to the brewery, and the storage capacity at the brewery. The brewery can process and store 4,000,000 pounds of malted grain per month. For each barrel of pale ale, the brewery uses 69.75 pounds of malted grain. For each barrel of porter, the brewery uses 85.25 pounds of malted grain.
Using this information, write and graph a linear inequality describing the total amount of malted grain used each month at the brewery.

Solution To write an inequality for the total amount of grain, we need to recognize that this brewery can use no more than 4,000,000 pounds of malted grain each month,

\[
\text{Total amount of malted grain } \leq 4,000,000 \text{ pounds}
\]

Malted grain is used in each of the two beers produced. The amount of grain used in the pale ale is found by multiplying the amount of grain used per barrel times the number of barrels of pale ale produced,

\[
\text{Amount of grain in pale ale: } 69.75 \text{ lbs/} \text{barrel} \cdot x_1 \text{ barrels} = 69.75x_1 \text{ lbs}
\]

The units in these two factors reduce to yield overall units of pounds.

We can also determine the amount of grain in the porter by writing

\[
\text{Amount of grain in porter: } 85.25 \text{ lbs/} \text{barrel} \cdot x_2 \text{ barrels} = 85.25x_2 \text{ lbs}
\]

Combining the amount of grain in the pale ale, \(69.75x_1\), and the amount of grain in the porter, \(85.25x_2\), yields the inequality for the grain usage,

\[
\frac{69.75x_1 + 85.25x_2}{\text{Total amount of grain used}} \leq 4,000,000
\]

To graph this inequality, change the inequality to an equal sign,

\[
69.75x_1 + 85.25x_2 = 4,000,000
\]

and find the intercepts:
\[
\begin{align*}
\text{Set } x_2 &= 0 \\
\text{Then} \\
69.75x_1 + 85.25(0) &= 4,000,000 \\
x_1 &= 4,000,000/69.75 \\
&\approx 57,348
\end{align*}
\]

\[
\begin{align*}
\text{Set } x_1 &= 0 \\
\text{Then} \\
69.75(0) + 85.25x_2 &= 4,000,000 \\
x_2 &= 4,000,000/85.25 \\
&\approx 46,921
\end{align*}
\]

\[
(x_1, x_2) = (57348, 0) \quad (x_1, x_2) = (0, 46921)
\]

As before, we'll graph this line with \( x_1 \) as the independent variable. We also use a solid line since the inequality includes an equal sign.

![Figure 8 - The border for the inequality. The border is drawn with a solid line since the line is included in the solution set.](image)

Pick a convenient test point like \((0, 0)\) to see which side of the line is in the solution set.

If we set \( x_1 = 0 \) and \( x_2 = 0 \) in the inequality, we get

\[
69.75(0) + 85.25(0) \leq 4,000,000
\]

\[
0 \leq 4,000,000 \quad \text{True}
\]
Since the test point at the origin makes the inequality true, shade the side of the line that includes the origin.

Figure 9 - The solution set for $69.75x_1 + 85.25x_2 \leq 4,000,000$ extends infinitely far to the left and below the line.

**Example 3** Find and Graph the Linear Inequality

Hops are also used to make the pale ale and porter styles of beer. Each barrel of pale ale requires 23.8 ounces of hops. Each barrel of porter requires 10.85 ounces of hops. The brewery is able to acquire 62,500 pounds of hops each month. If $x_1$ barrels of pale ale are produced, and $x_2$ barrels of porter are produced, find and graph the inequality describing the total amount of hops used each month.

**Solution** This example is very similar to 0. The main difference is the different units in the problem. In this example we’ll derive the inequalities in terms of ounces.
To convert the total amount of hops to ounces, we would calculate

\[
62,500 \text{ pounds} \cdot \frac{16 \text{ ounces}}{1 \text{ pound}} = 1,000,000 \text{ ounces}.
\]

Now the total amount of hops used must satisfy

\[
\text{Total amount of hops} \leq 1,000,000.
\]

The total amount of hops is the sum of the amount of hops in each beer. The amount of hops in \( x_1 \) barrels of pale is

\[
\text{Ounces of hops in pale ale} = \frac{23.8 \text{ ounces}}{1 \text{ barrel}} \cdot x_1 \text{ barrels} = 23.8x_1 \text{ ounces}
\]

The amount of hops in \( x_2 \) barrels of porter is

\[
\text{Ounces of hops in porter} = \frac{10.85 \text{ ounces}}{1 \text{ barrel}} \cdot x_2 \text{ barrels} = 10.85x_2 \text{ ounces}
\]
In each of these expressions, the units reduce to give overall units of ounces. The sum of the amount of hops in each beer must be less than or equal to 1,000,000 ounces or

\[ 23.8x_1 + 10.85x_2 \leq 1,000,000. \]

To graph the inequality, we graph the border of the solution set defined by the equation \( 23.8x_1 + 10.85x_2 = 1,000,000 \). The easiest way to graph this equation is to find the intercepts of the graph.

<table>
<thead>
<tr>
<th>( x_1 ) - Intercept</th>
<th>( x_2 ) - Intercept</th>
</tr>
</thead>
<tbody>
<tr>
<td>Set ( x_2 = 0 )</td>
<td>Set ( x_1 = 0 )</td>
</tr>
<tr>
<td>Then</td>
<td>Then</td>
</tr>
<tr>
<td>( 23.8x_1 + 10.85(0) = 1,000,000 )</td>
<td>( 23.8(0) + 10.85x_2 = 1,000,000 )</td>
</tr>
<tr>
<td>( 23.8x_1 = 1,000,000 )</td>
<td>( 10.85x_2 = 1,000,000 )</td>
</tr>
<tr>
<td>( x_1 \approx 42,017 )</td>
<td>( x_2 \approx 92,166 )</td>
</tr>
<tr>
<td>( (x_1, x_2) = (42017, 0) )</td>
<td>( (x_1, x_2) = (0, 92166) )</td>
</tr>
</tbody>
</table>

The intercepts of the corresponding equations are approximately \((0, 92166)\) and \((42017, 0)\).

By testing the inequality \( 23.8x_1 + 10.85x_2 \leq 1,000,000 \) at the origin,

\[ 23.8(0) + 10.85(0) \leq 1,000,000 \quad \text{True} \]

we see that the side of the line including \((0,0)\) must be shaded. This gives the solution set shown below.
Figure 11 - The solution set for the hops inequality lies to the left of $23.8x_1 + 10.85x_2 = 1,000,000$. 
Question 2: How do you graph a system of linear inequalities?

A craft brewery cannot produce unlimited amount of beer each month. As we have seen in this section, there are constraints on the amount of beer that can be fermented as well as the amounts of each ingredient that can be shipped to the brewery and stored on site. If \( x_1 \) barrels of pale ale and \( x_2 \) barrels are produced per month,

\[
\begin{align*}
    x_1 + x_2 & \leq 50,000 & \text{Capacity Constraint} \\
    69.75x_1 + 85.25x_2 & \leq 4,000,000 & \text{Malt Constraint} \\
    23.8x_1 + 10.85x_2 & \leq 1,000,000 & \text{Hops Constraint}
\end{align*}
\]

Together these inequalities define possible combinations that the brewery may produce. In addition, it does not make sense for the amount of beer produced to be negative. So in addition to these inequalities, we also require that

\[
x_1 \geq 0, \quad x_2 \geq 0
\]

Together these inequalities form a system of linear inequalities.

A system of linear inequalities is two or more linear inequalities solved simultaneously. Each linear inequality has a solution that is a half plane. The solution set of the system of linear inequalities is all ordered pairs that make all inequalities in the system true. If we examine the solutions sets of each of the individual inequalities together, the solution set of the system of inequalities is where all of the individual solution sets overlap.

**The Solution to a System of Linear Inequalities**

1. Graph the corresponding linear equation for each of the linear inequalities. If the inequality includes an equal sign, graph the equation with a solid line. If the inequality does not include an equals, graph the equation with a dashed line.
2. For each inequality, use a test point to determine which side of the line is in the solution set. Instead of using shading to indicate the solution, use arrows along the line pointing in the direction of the solution.

3. The solution to the system of linear inequalities is all areas on the graph that are in the solution of all of the inequalities. Shade any areas on the graph that the arrows you drew indicate are in common.

**Example 4**  **Graph the System of Linear Inequalities**

Graph the solution set for the system of linear inequalities

\[
 y < -2x + 5 \\
 x \geq 0 \\
 y \geq 0
\]

**Solution** To get started, we need to graph the equations \( y = -2x + 5 \), \( x = 0 \), and \( y = 0 \). The first equation will be graphed as a dashed line since the corresponding inequality is a strict inequality. The vertical line \( x = 0 \) and horizontal line \( y = 0 \) will be graphed as solid lines since the corresponding inequalities include an equal sign.
With these boundary lines in place, we need to test a point in each inequality to know the individual solution for each inequality. It is not possible to use \((0,0)\) since it lies on the horizontal and vertical lines.

Another point that is easy to test is \((1,1)\).

<table>
<thead>
<tr>
<th>Inequality</th>
<th>Set (x=1) and (y=1)</th>
<th>True or False?</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y&lt;-2x+5)</td>
<td>(1&lt;-2(1)+5)</td>
<td>True</td>
</tr>
<tr>
<td>(x \geq 0)</td>
<td>(1 \geq 0)</td>
<td>True</td>
</tr>
<tr>
<td>(y \geq 0)</td>
<td>(1 \geq 0)</td>
<td>True</td>
</tr>
</tbody>
</table>

For each line, place arrows on the lines indicating which side is a part of the solution. Since the test point is true in each inequality, we need to place arrows on each line pointing toward the test point.
Any point in the triangular region where (1,1) lies will make all of the inequalities true. In any other part of the graph, one or two of the inequalities might be true, but not all three in the system. Shade the triangular region where all of the individual solutions overlap to give the graph in Figure 14.
When the solution set to a system of inequalities can be enclosed in a circle, the solution is bounded. The solution in Example 4 is an example of a bounded solution set.

Example 5  **Graph the System of Linear Inequalities**

Graph the solution set for the system of inequalities

\[
\begin{align*}
  x_2 & \geq -\frac{2}{3}x_1 + 3 \\
  5x_1 + 3x_2 & \geq 18 \\
  x_1 & \geq 0 \\
  x_2 & \geq 0
\end{align*}
\]

**Solution** In this example, subscripted variables are used. This makes it harder to pick an independent variable. Either variable can be the independent variable. For no particular reason, we’ll graph $x_1$ on the horizontal axis and $x_2$ on the vertical axis.

This system includes the non-negativity constraints $x_1 \geq 0$ and $x_2 \geq 0$. These tell us that the solution must lie in a quadrant where all variables are not negative. We can simplify the example by realizing that this is in the first quadrant. By graphing all of the lines primarily in the first quadrant, we can speed up the graphing process.

To graph the other two constraints, we need to graph the lines that form the border of the solution set,

\[
\begin{align*}
  x_2 &= -\frac{2}{3}x_1 + 3 \\
  5x_1 + 3x_2 &= 18
\end{align*}
\]
The first equation is in slope-intercept form with a slope of \(-\frac{2}{3}\) and a vertical intercept of 3. Drawing a line through \((0,3)\) with a slope of \(-\frac{2}{3}\) gives the border of the first constraint.

The second line is easy to graph if we find the intercepts. If we set \(x_1 = 0\), we get \(x_2 = 6\) yielding the vertical intercept \((x_1, x_2) = (0, 6)\). The other intercept is located by setting \(x_2 = 0\). The resulting equation can be solved for \(x_1\) to yield the horizontal intercept \((x_1, x_2) = \left(\frac{18}{5}, 0\right)\).

![Figure 15 - The borders of the first two inequalities in Example 5. The graph is drawn in the first quadrant due to non-negativity constraints.](image)

Drawing a line through these two ordered pairs results in the border for the second constraint. Both lines must be drawn with a solid line since both constraints include an equal sign in the inequality.
Now pick a test point like \((0,0)\) to determine which side of each line should be shaded.

<table>
<thead>
<tr>
<th>Inequality</th>
<th>Set (x_1 = 0) and (x_2 = 0)</th>
<th>True or False?</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_2 \geq -\frac{2}{3}x_1 + 3)</td>
<td>(0 \geq -\frac{2}{3}(0) + 3)</td>
<td>False</td>
</tr>
<tr>
<td>(5x_1 + 3x_2 \geq 18)</td>
<td>(5(0) + 3(0) \geq 18)</td>
<td>False</td>
</tr>
</tbody>
</table>

Each inequality is false at the test point so the solution set to each inequality must be on the side of the line where the test point is not located. Use arrows on each line to indicate the side of each line where the test point is not located.

The solution set to the system is all of the ordered pairs in the first quadrant (due to the non-negativity constraints) that are solutions to each of the inequalities in the system. By noting where the solution sets for each inequality overlap, we get the solution set below.
Figure 17 – The solution set to the system in Example 5.

Because this solution set extends infinitely far, the region cannot be enclosed in a circle. Solution sets that cannot be enclosed in a circle are called unbounded solution sets.

**Example 6  Graph the System of Linear Inequalities**

Graph the solution set for the system of inequalities

\[ y \geq x + 3 \]
\[ x - y > 0 \]

**Solution** The border of the solution set is formed by

\[ y = x + 3 \]
\[ x - y = 0 \]

The first equation is a line in slope-intercept form with a slope of 1 and a vertical intercept of 3. This line is graphed as a solid line since an equal sign is part of the inequality.
The second equation can be solved for $y$ to yield $y = x$. This is a line with a slope of 1 that passes through the origin. Since the inequality is a strict inequality, the border is graphed with a dashed line.

![Graph of inequalities]

Figure 18 - The borders of the inequalities in Example 6.

The border of the second inequality passes through the ordered pairs $(0,0)$ and $(1,1)$, the ordered pairs we have used in other examples as test points. For this example, we'll use $(1,0)$ as the test point.

<table>
<thead>
<tr>
<th>Inequality</th>
<th>Set $x = 1$ and $y = 0$</th>
<th>True or False?</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y \geq x + 3$</td>
<td>$0 \geq 1 + 3$</td>
<td>False</td>
</tr>
<tr>
<td>$x - y &gt; 0$</td>
<td>$1 - 0 &gt; 0$</td>
<td>True</td>
</tr>
</tbody>
</table>
Figure 19 - The borders of the inequalities with arrows indicating the individual inequality solution. The first inequality is false at the test point so the side opposite is the solution. For the second inequality, the inequality is true at the test point so that side is the solution for the inequality.

As with previous examples, the solution set of the system of inequalities is where all of the individual inequality solution sets overlap. Examining the graph carefully, we note that there is no place on the graph where all of the inequalities overlap. There are no ordered pairs that satisfy all of the inequalities simultaneously, so there are no solutions to the system of inequalities.

Now that we’ve tried out the strategy for solving several simple systems of inequalities, let’s revisit the system of inequalities for the craft brewery.
Example 7  Graph the System of Inequalities for the Craft Brewery

The system of inequalities for a craft brewery is

\[
\begin{align*}
    x_1 + x_2 & \leq 50,000 & \text{Capacity Constraint} \\
    69.75x_1 + 85.25x_2 & \leq 4,000,000 & \text{Malt Constraint} \\
    23.8x_1 + 10.85x_2 & \leq 1,000,000 & \text{Hops Constraint} \\
    x_1 & \geq 0 \\
    x_2 & \geq 0
\end{align*}
\]

where \( x_1 \) is the number of barrels of pale ale produced, and \( x_2 \) is the number of barrels of porter produced. Graph the system of inequalities with \( x_1 \) as the independent variable.

**Solution**  The non-negativity constraints restrict the solution to the first quadrant. Because of this, we'll restrict the borders of the other three inequalities to this quadrant and not extend any shading of the final solution beyond the first quadrant.

The borders of the inequalities for capacity, malt, and hops are given by the equations

\[
\begin{align*}
    x_1 + x_2 &= 50,000 \\
    69.75x_1 + 85.25x_2 &= 4,000,000 \\
    23.8x_1 + 10.85x_2 &= 1,000,000
\end{align*}
\]
Each of these equations was graphed earlier in this section. Putting all of the lines on a single graph yields the graph in Error! Reference source not found.

To find the solution sets of each of the inequalities, let's test the ordered pair \((0,0)\) in each inequality and indicate shading with arrows in the graph.

<table>
<thead>
<tr>
<th>Inequality</th>
<th>Set (x_1 = 0) and (x_2 = 0)</th>
<th>True or False?</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_1 + x_2 \leq 50,000)</td>
<td>(0 + 0 \leq 50,000)</td>
<td>True</td>
</tr>
<tr>
<td>(69.75x_1 + 85.25x_2 \leq 4,000,000)</td>
<td>(69.75(0) + 85.25(0) \leq 4,000,000)</td>
<td>True</td>
</tr>
<tr>
<td>(23.8x_1 + 10.85x_2 \leq 1,000,000)</td>
<td>(23.8(0) + 10.85(0) \leq 1,000,000)</td>
<td>True</td>
</tr>
</tbody>
</table>

Based on this table, the test point must be included in the solution set of each of the inequalities. The arrows on each of the lines must point to the half-plane containing the test point for each inequality.
Figure 21 - For each inequality, the test point must be included in the solution suggesting the shading on the graph.

The individual solution sets all overlap in a region that is to the left and below all of the lines.

Figure 22 - The region shaded in gray is the solution to the system of inequalities for a craft brewery. This region is where the solutions to all inequalities in the system coincide.
At any point in the solution set, the brewery can produce the number of barrels of beer indicated by the ordered pair and satisfy all of the constraints for capacity, malt and hops.