Question 2: How do you solve a linear programming problem with a graph?

Now that we have several linear programming problems, let's look at how we can solve them using the graph of the system of inequalities. The linear programming problem for the craft brewery was found to be

Maximize $P = 100x_1 + 80x_2$
subject to

\[
\begin{align*}
    x_1 + x_2 &\leq 50,000 \\
    69.75x_1 + 85.25x_2 &\leq 4,000,000 \\
    23.8x_1 + 10.85x_2 &\leq 1,000,000 \\
    x_i &\geq 0, \ x_2 &\geq 0
\end{align*}
\]

Figure 2 - The feasible region for the craft brewery problem.

The shaded region corresponds to all of the possible combinations of pale ale and porter that satisfy the constraints. Since the solution to the maximization of profit must come from this region, it is called the feasible region. This means that the ordered pairs in the shaded region are feasible solutions for the linear programming problem.
To solve the linear programming problem, we need to find which combination of $x_1$ and $x_2$ lead to the greatest profit. We could pick possible combination from the graph and calculate the profit at each location on the graph, but this would be extremely time consuming. Instead we'll pick a value for the profit and find all of the ordered pairs on the graph that match that profit.

Suppose we start with a profit of $3,000,000. Substitute this value into the objective function to yield the equation

$$3,000,000 = 100x_1 + 80x_2.$$ 

If we graph this line on the same graph as the system of inequalities, we get the dashed line labeled $P = 3,000,000$. A line on which the profit is constant is called an isoprofit line. The prefix iso- means same so that an isoprofit line has the same profit along it. Along the isoprofit line $P = 3,000,000$, every combination of pale ale and porter leads to a profit of $3,000,000$.

Figure 3 - Several levels of profit at $3,000,000$, $4,000,000$ and $5,000,000$. 
The isoprofit lines $P = 4,000,000$ and $P = 5,000,000$ can be graphed in the same manner and are pictured in Figure 3.

As profit increases, the isoprofit lines move farther to the right. Higher profit levels lead to similar lines that are farther and farther from the origin. Eventually the isoprofit line is outside of the feasible region. For equally spaced profit levels we get equally spaced parallel lines on the graph. Notice that the $P = 5,000,000$ is completely outside the shaded region. This means that no combination of pale ale and porter will satisfy the inequalities and earn a profit of $5,000,000$. There is an isoprofit line, $P \approx 4,706,564$, that will just graze the feasible region. This isoprofit line will touch where the border for the capacity constraint and the border of the hops constraint intersect. Points where the borders for the constraints cross are called corner points of the feasible region.

![Figure 4](image)

Figure 4 - The optimal production level for the craft brewery rounded to one decimal place. The isoprofit for this level is also graphed.

If the profit is any higher than this level, the isoprofit line no longer contains any ordered pairs from the feasible region. This means that this profit level is the maximum profit and it occurs at the corner point where 35,328.2 barrels of pale ale and 14,671.8 barrels of porter are produced.
Two of the constraint borders intersect at the corner point corresponding to the optimal solution. These constraints, the constraints for capacity and hops, are said to be binding constraints. The malt constraint does not intersect the corner point that maximizes profit so it is a nonbinding constraint.

For resources corresponding to binding constraints, all of the resources are used. In this case, all of the capacity and hops are used to produce the optimal amounts of beer. However, all of the malt is not used which is why the malt constraint is not binding at the optimal solution.

The fact that the optimal solution occurs at a corner point of the feasible region suggests the following insight.

The optimal solution to a linear programming occurs at a corner point to the feasible region or along a line connecting two adjacent corner points of the feasible region.

If a feasible region is bounded, there will always be an optimal solution. Unbounded feasible regions may or may not have an optimal solution.

We can use this insight to develop the following strategy for solving linear programming problems with two decision variables.

1. Graph the feasible region using the system of inequalities in the linear programming problem.

2. Find the corner points of the feasible region.

3. At each corner point, find the value of the objective function.
By examining the value of the objective function, we can find the maximum or minimum values. If the feasible region is bounded, the maximum and minimum values of the objective function will occur at one or more of the corner points. If two adjacent corner points lead to same maximum (or minimum) value, then the maximum (or minimum) value also occurs at all points on the line connecting the adjacent corner points.

Unbounded feasible regions may or may not have optimal values. However, if the feasible region is in the first quadrant and the coefficients of the objective function are positive, then there is a minimum value at one or more of the corner points. There is no maximum value in this situation. Like a bounded region, if the minimum occurs at two adjacent corner points, it also occurs on the line connecting the adjacent corner points.

Figure 5 – The feasible region in Graph (a) can be enclosed in a circle so it is a bounded feasible region. The feasible region in Graph (b) extends infinitely far to the upper right so it cannot be enclosed in a circle. This feasible region is unbounded.
Example 2  Solve the Linear Programming Problem Graphically

Solve the linear programming problem graphically:

Maximize \( z = 5x + 6y \)
subject to
\[
\begin{align*}
2x + y & \leq 4 \\
x + 2y & \leq 4 \\
x & \geq 0, \ y & \geq 0
\end{align*}
\]

Solution  The feasible region is in the first quadrant since the non-negativity constraints \( x \geq 0 \) and \( y \geq 0 \) are included in the system of inequalities.

The other two inequalities are converted to equations and graphed using the horizontal and vertical intercepts.

<table>
<thead>
<tr>
<th>Equation</th>
<th>Find Intercepts</th>
<th>Coordinates of Intercepts</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2x + y = 4 )</td>
<td>( x = 0 \Rightarrow 2(0) + y = 4 ) ( y = 4 ) ( y = 0 \Rightarrow 2x + 0 = 4 ) ( x = 2 )</td>
<td>(0,4) and (2,0)</td>
</tr>
<tr>
<td>( x + 2y = 4 )</td>
<td>( x = 0 \Rightarrow 0 + 2y = 4 ) ( y = 2 ) ( y = 0 \Rightarrow x + 2(0) = 4 ) ( x = 4 )</td>
<td>(0,2) and (4,0)</td>
</tr>
</tbody>
</table>

We can graph these intercepts in the first quadrant and connect them with solid lines to yield the following graph.
To determine what portion of this graph corresponds to the feasible region, we need to test the original inequalities with a point from the graph. Although we could choose any ordered pair that is not on one of the lines, in this example we’ll test with (1,1).

<table>
<thead>
<tr>
<th>Inequality</th>
<th>Set ( x = 1 ) and ( y = 1 )</th>
<th>True or False?</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2x + y \leq +4 )</td>
<td>( 2(1)+1 \leq 4 )</td>
<td>True</td>
</tr>
<tr>
<td>( x + 2y \leq 4 )</td>
<td>( 1+2(1) \leq 4 )</td>
<td>True</td>
</tr>
</tbody>
</table>

The test point is in the solution set of each inequality. Sketch arrows on each line pointing in the direction of the test point.
Figure 7 - Since both inequalities are true, the arrows on each line should point in the direction of the test point.

The solution sets for the inequalities overlap in the lower left hand corner of the graph.

Figure 8 - The feasible region for the linear programming problem in Example 2.
The optimal solution for the linear programming problem must come from this region. Since this is a bounded feasible region, the maximum will occur at one of the corner points. From the graph and the intercepts of the borders, we can see easily identify three corner points: \((0,0)\), \((2,0)\), and \((0,3)\). One more corner point is located at the intersection of the line \(2x + y = 4\) and \(x + 2y = 4\). We could find this corner point by the Substitution Method, Elimination Method or with matrices. In this example we’ll use the Substitution Method since it is easy to solve for a variable in one of the equations.

To apply the Substitution Method to the system of equations

\[
\begin{align*}
2x + y &= 4 \\
x + 2y &= 4
\end{align*}
\]

solve the first equation for \(y\). If we subtract 2x form both sides of the first equation we get \(y = -2x + 4\).

Replace \(y\) with \(-2x + 4\) in the second equation and solve for \(x\),

\[
\begin{align*}
x + 2(-2x + 4) &= 4 & \text{Replace } y \text{ with } -2x + 4 \text{ in the second equation} \\
x - 4x + 8 &= 4 & \text{Remove parentheses} \\
-3x + 8 &= 4 & \text{Combine like terms} \\
-3x &= -4 & \text{Subtract 8 from both sides} \\
x &= \frac{4}{3} & \text{Divide both sides by -3}
\end{align*}
\]

To find \(y\), substitute the value for \(x\) into \(y = -2x + 4\) to yield

\[
y = -2\left(\frac{4}{3}\right) + 4 = \frac{4}{3}
\]

The solution to the system is \(x = \frac{4}{3}\) and \(y = \frac{4}{3}\).
Let's label the corner points on the graph and calculate the value of the objective function at each of them.

![Graph](image)

**Figure 9 - The feasible region with the corner points labeled.**

<table>
<thead>
<tr>
<th>Corner Point ((x, y))</th>
<th>(z = 5x + 6y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 0))</td>
<td>(z = 5(0) + 6(0) = 0)</td>
</tr>
<tr>
<td>((2, 0))</td>
<td>(z = 5(2) + 6(0) = 10)</td>
</tr>
<tr>
<td>((0, 2))</td>
<td>(z = 5(0) + 6(2) = 12)</td>
</tr>
<tr>
<td>((\frac{4}{3}, \frac{4}{3}))</td>
<td>(z = 5(\frac{4}{3}) + 6(\frac{4}{3}) = \frac{44}{3})</td>
</tr>
</tbody>
</table>

Since \(\frac{44}{3}\) is approximately 14.67, the corner point at \((\frac{4}{3}, \frac{4}{3})\) yields the maximum value of the objective function.
Example 3  Solve the Linear Programming Problem Graphically

Solve the linear programming problem graphically:

Minimize $w = 4y_1 + y_2$

subject to

$y_2 \geq -\frac{1}{4}y_1 + 2$

$7y_1 + 4y_2 \geq 32$

$y_1 \geq 0, y_2 \geq 0$

Solution To graph the system of inequalities, we need to pick an independent variable to graph on the horizontal axis. Although the choice is arbitrary, in this example we'll graph $y_1$ on the horizontal axis and $y_2$ on the vertical axis.

This linear programming problem includes non-negativity constraints, $y_1 \geq 0$ and $y_2 \geq 0$. In the graph, we'll account for these constraints by graphing the system of inequalities in the first quadrant only.

The borders for the other two constraints are graphed by utilizing the slope-intercept form of a line and intercepts.

$y_2 = -\frac{1}{4}y_1 + 2$

slope = $-\frac{1}{4}$

vertical intercept = 2

$7y_1 + 4y_2 = 32$

$y_1 = 0$  $\Rightarrow$  $y_2 = 8$

$y_2 = 0$  $\Rightarrow$  $y_1 = \frac{12}{7}$
Notice that the borders are drawn with solid lines since the inequalities are not strict inequalities. To shade the solution on the graph, substitute the test point \((0,0)\) into each inequality.

<table>
<thead>
<tr>
<th>Inequality</th>
<th>Set ( y_1 = 0 ) and ( y_2 = 0 )</th>
<th>True or False?</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y_2 \geq -\frac{1}{4} y_1 + 2 )</td>
<td>( 0 \geq -\frac{1}{4}(0) + 2 )</td>
<td>False</td>
</tr>
<tr>
<td>( 7y_1 + 4y_2 \geq 32 )</td>
<td>( 7(0) + 4(0) \geq 32 )</td>
<td>False</td>
</tr>
</tbody>
</table>

Since each inequality is false, draw arrows on each line pointing away from the test point.
Figure 11 - Since each inequality is false at (0, 0), the side opposite each test point must be shaded.

The shading for each inequality coincides in the upper right portion of the graph.

Figure 12 - The solution set for the system of inequalities for Example 3.
The feasible region extend infinitely far towards the upper right. This feasible region is unbounded since it cannot be enclosed in a circle. A minimum value may or may not exist.

Three corner points are evident in the feasible region. Two of the corner points, \((0,8)\) and \((8,0)\), are on the lines drawn earlier. The third corner point occurs where the line \(y_2 = \frac{1}{4}y_1 + 2\) and the line \(7y_1 + 4y_2 = 32\) cross.

![Figure 13 – The corner points for the feasible region in 0.](image)

Since the first equation, \(y_2 = \frac{1}{4}y_1 + 2\), is solved for \(y_2\), it is easiest to use the Substitution Method to solve for the intersection point.

Substitute \(y_2 = \frac{1}{4}y_1 + 2\) into \(7y_1 + 4y_2 = 32\) to yield

\[
7y_1 + 4\left(\frac{1}{4}y_1 + 2\right) = 32
\]

Solve this equation by isolating \(y_1\):
\[
7y_1 - y_1 + 8 = 32 \\
6y_1 + 8 = 32 \\
6y_1 = 24 \\
y_1 = 4
\]

Remove the parentheses  
Combine like terms  
Subtract 8 from each side  
Divide both sides by 6

Put this value into the first equation to give \( y_2 = -\frac{1}{4}(4) + 2 = 1 \). So the third corner point is located at \((y_1, y_2) = (4,1)\).

To find the minimum value of the objective function \( w = 4y_1 + y_2 \), substitute the corner points.

<table>
<thead>
<tr>
<th>Corner Point ((y_1, y_2))</th>
<th>(w = 4y_1 + y_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0,8))</td>
<td>(w = 4(0) + 8 = 8)</td>
</tr>
<tr>
<td>((4,1))</td>
<td>(w = 4(4) + 1 = 17)</td>
</tr>
<tr>
<td>((8,0))</td>
<td>(w = 4(8) + 0 = 32)</td>
</tr>
</tbody>
</table>
It appears that the minimum is at the ordered pair $(y_1, y_2) = (0,8)$.

To confirm that the minimum is located at $(0,8)$, look at three isolevels of $w = 4$, $w = 8$, and $w = 12$. The level $w = 8$ should pass through the corner point, but where do the other levels lie?

If we set $w$ equal to each of these levels in the objective function and add these to the graphs, we see how the isolevels behave near the feasible region.

As anticipated, the $w = 8$ level passes through the corner point at $(0,8)$. Higher $w$ levels, like $w = 12$, pass through the feasible region. As $w$ increases, the isolevel lines move farther and farther into the feasible region meaning the feasible region has no maximum value.

We are interested in a minimum value for the objective function. The lowest value of $w$ that includes any points in the feasible region is $w = 8$. Any levels lower than $w = 8$, like $w = 4$, lie completely outside of the feasible region. The level $w = 8$ passes through the corner point $(8,0)$ and is the optimal value for the linear programming problem.
In each of the previous examples, there was one optimal solution. In the next example, there are many possible solutions to the linear programming problem. Each solution yields the same value of the objective function.

**Example 4  Solve the Linear Programming Problem Graphically**

Solve the linear programming problem graphically:

Maximize $z = 5x + 15y$

subject to

$-x + 6y \leq 30$

$x + 3y \leq 24$

$x \leq 8$

$x \geq 0, y \geq 0$

**Solution** As with the earlier examples, the non-negativity constraints allow us to graph the system of inequalities in the first quadrant only. To graph the borders of the other three constraints, convert each of the inequalities to equations,

$-x + 6y = 30$

$x + 3y = 24$

$x = 8$
The line \( x = 8 \) is a vertical line passing through the horizontal axis at \((8,0)\). We can place these three lines on a graph as shown in Figure 15.

<table>
<thead>
<tr>
<th>Equation</th>
<th>Put in Slope-Intercept Form</th>
<th>Slope and Vertical Intercept</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-x + 6y = 30)</td>
<td>(6y = x + 30) Add (x) to both sides</td>
<td>(\text{slope} = \frac{1}{6})</td>
</tr>
<tr>
<td></td>
<td>(y = \frac{1}{6}x + \frac{30}{6}) Divide both sides by 6</td>
<td>(\text{intercept} = (0,5))</td>
</tr>
<tr>
<td></td>
<td>(y = \frac{1}{6}x + 5) Simplify</td>
<td></td>
</tr>
<tr>
<td>(x + 3y = 24)</td>
<td>(3y = -x + 24) Subtract (x) from both sides</td>
<td>(\text{slope} = -\frac{1}{3})</td>
</tr>
<tr>
<td></td>
<td>(y = -\frac{1}{3}x + \frac{24}{3}) Divide both sides by 3</td>
<td>(\text{intercept} = (0,8))</td>
</tr>
<tr>
<td></td>
<td>(y = -\frac{1}{3}x + 8) Simplify</td>
<td></td>
</tr>
</tbody>
</table>

Figure 15 - A graph of the border of each inequality in 1.1Question 1Example 4.

To shade the solution on the graph, substitute the test point \((0,0)\) into each inequality.
Inequality | Set $x = 0$ and $y = 0$ | True or False?
--- | --- | ---
$-x + 6y \leq 30$ | $-0 + 6(0) \leq 30$ | True
$x + 3y \leq 24$ | $0 + 3(0) \leq 24$ | True
$x \leq 8$ | $0 \leq 8$ | True

Figure 16 - Since each inequality is true, shade the half plane formed by each inequality where the test point lies.

The individual solution sets all overlap in the feasible region shown below.
Since this feasible region is bounded, the maximum value of the objective function is located at one of the corner points or along a line connecting adjacent corner points. Several of the corner points are easily identified from the intercepts. However, the two corner points in the upper-right corner of the feasible region are not so obvious.

The red border, given by the line $x + 3y = 24$, and the green line, given by $x = 8$, intersect at one of these corner points.

If we substitute $x = 8$ into $x + 3y = 24$, we can solve for $y$ to get the first corner point:

\[
\begin{align*}
8 + 3y &= 24 \\
3y &= 16 \\
y &= \frac{16}{3}
\end{align*}
\]

The corner point is at \( (8, \frac{16}{3}) \).
Another corner point is where red line, $x + 3y = 24$, intersects the blue line, $-x + 6y = 30$. Using the Substitution Method, we can solve for $x$ in the first equation to yield $x = -3y + 24$. Substitute the expression $-3y + 24$ for $x$ in the second equation and solve for $y$:

\[
-(-3y + 24) + 6y = 30 \\
9y - 24 = 30 \\
9y = 54 \\
y = 6
\]

Substitute $-3y + 24$ for $x$ Remove the parentheses and combine like terms Add 24 to both sides Divide both sides by 9

If we set $y = 6$ in $x = -3y + 24$, we get the corresponding $x$ value, $x = 6$
Figure 19 - The feasible region with a corner points at (6, 6) and \( \left( \frac{8}{3}, 6 \right) \) shown.

Now that we have the corner points, we can substitute the coordinates into the objective function to find the maximum value.

Figure 20 - The corner points and the corresponding values of the objective function.
<table>
<thead>
<tr>
<th>Corner Point ((x, y))</th>
<th>(z = 5x + 15y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0,0))</td>
<td>(z = 5(0) + 15(0) = 0)</td>
</tr>
<tr>
<td>((0,5))</td>
<td>(z = 5(0) + 15(5) = 75)</td>
</tr>
<tr>
<td>((8,0))</td>
<td>(z = 5(8) + 15(0) = 40)</td>
</tr>
<tr>
<td>((6,6))</td>
<td>(z = 5(6) + 15(6) = 120)</td>
</tr>
<tr>
<td>((8, \frac{16}{3}))</td>
<td>(z = 5(8) + 15(\frac{16}{3}) = 120)</td>
</tr>
</tbody>
</table>

Two adjacent corner points yield the same value, \(z = 120\). This means that the objective function is maximized at any point of a line connecting \((6,6)\) and \((8, \frac{16}{3})\).

Figure 21 - The isolevel \(z = 120\) passes through two adjacent corner points indicating that the objective function is optimized anywhere on the line connecting the points.

If we graph the isolevel \(5x + 15y = 120\) on the feasible region, we see that it passes through these points.
For any higher isolevel, the line would be outside of the feasible region. Therefore, this level is the maximum value of the objective function.

Now that we have established the strategy for solving linear programming problems graphically, let’s return to the two brewing examples.

**Example 5  Solve the Linear Programming Application Graphically**

The linear programming problem for a craft brewery is

Maximize \( P = 100x_1 + 80x_2 \)

subject to

\[
\begin{align*}
50,000 & \leq 69.75x_1 + 85.25x_2 \\
69.75x_1 + 85.25x_2 & \leq 4,000,000 \\
23.8x_1 + 10.85x_2 & \leq 1,000,000 \\
x_1 & \geq 0, \ x_2 \geq 0
\end{align*}
\]

where \( x_1 \) is the number of barrels of pale ale and \( x_2 \) is the number of barrels of porter produced. Solve this linear programming problem graphically to find the production level that maximizes the profit \( P \).

**Solution** In earlier examples, we developed the system of inequalities and objective function for a craft brewery. By graphing many isoprofit levels we were able to find the optimal value for the profit function.

In this example we’ll use the simpler strategy we developed for maximizing the objective function. We’ll need to find the corner points of the feasible region and evaluate them in the objective function to find the maximum profit.
The feasible region has five corner points that we must find. The easiest corner points to locate are the origin, the vertical intercept for $69.75x_1 + 85.25x_2 = 4,000,000$ and the horizontal intercept for $23.8x_1 + 10.85x_2 = 1,000,000$.

To find the vertical intercept for $69.75x_1 + 85.25x_2 = 4,000,000$, set $x_1 = 0$ and solve for $x_2$:

$$69.75(0) + 85.25x_2 = 4,000,000$$

Set $x_1 = 0$

$$85.25x_2 = 4,000,000$$

$$x_2 = \frac{4,000,000}{85.25} \approx 46,920.8$$

Divide both sides by 85.25

The coordinates of this vertical intercept are approximately $(x_1, x_2) = (0, 46920.8)$.

To find the horizontal intercept for $23.8x_1 + 10.85x_2 = 1,000,000$, set $x_2 = 0$ and solve for $x_1$:
\[ 23.8x_1 + 10.85(0) = 1,000,000 \]
\[ 23.8x_1 = 1,000,000 \]
\[ x_1 = \frac{1,000,000}{23.8} \approx 42,016.8 \]

Set \( x_2 = 0 \)

The coordinates of this horizontal intercept are approximately \((x_1, x_2) = (42016.8, 0)\).

The other two corner points are points of intersection between the different borders. To find the corner point where the line \(x_1 + x_2 = 50,000\) and the line \(69.75x_1 + 85.25x_2 = 4,000,000\) intersect, write this system of equations as the augmented matrix

\[
\begin{bmatrix}
1 & 1 & 50,000 \\
69.75 & 85.25 & 4,000,000
\end{bmatrix}
\]

We can use a graphing calculator or row operations to put this matrix into its reduced row echelon. The reduced row echelon form is

\[
\begin{bmatrix}
1 & 0 & 16,935.5 \\
0 & 1 & 33,064.5
\end{bmatrix}
\]

where the last column has been rounded to one decimal place. This places the corner point at approximately \((16,935.5, 33,064.5)\).

The last corner point is located where the line \(x_1 + x_2 = 50,000\) and the line \(23.8x_1 + 10.85x_2 = 1,000,000\) intersect. The augmented matrix for this system is

\[
\begin{bmatrix}
1 & 1 & 50,000 \\
23.8 & 10.85 & 1,000,000
\end{bmatrix}
\]

The reduced row echelon form for this matrix is
The third column has been rounded to one decimal place. This corresponds to the corner point \((35,328.2, 14,671.8)\).

To find the production level that results in maximum profit, we need to substitute the corner points into the profit function.

<table>
<thead>
<tr>
<th>Approximate Corner Point ((x_1, x_2))</th>
<th>(P = 100x_1 + 80x_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0,0))</td>
<td>(P = 100(0) + 80(0) = 0)</td>
</tr>
<tr>
<td>((0, 46,920.8))</td>
<td>(P = 100(0) + 80(46,920.8) \approx 3,753,663)</td>
</tr>
<tr>
<td>((42,016.8, 0))</td>
<td>(P = 100(42,016.8) + 80(0) \approx 4,201,680)</td>
</tr>
<tr>
<td>((16,935.5, 33,064.5))</td>
<td>(P = 100(16,935.5) + 80(33,064.5) \approx 4,338,710)</td>
</tr>
<tr>
<td>((35,328.2, 14,671.8))</td>
<td>(P = 100(35,328.2) + 80(14,671.8) \approx 4,706,564)</td>
</tr>
</tbody>
</table>

The maximum profit is approximately $4,706,564 and is achieved when 35,328.2 barrels of pale ale and 14,671.8 barrels of porter are produced. The term “approximately” is used since the production levels have been rounded to the nearest tenth of a barrel.
Example 6  Solve the Linear Programming Application Graphically

The linear programming problem for the contract breweries is

Minimize \( C = 100Q_1 + 125Q_2 \)

subject to

\[
\begin{align*}
Q_1 + Q_2 & \geq 10,000 \\
Q_2 & \geq 0.25Q_1 \\
Q_2 & \leq Q_1 \\
Q_1 & \geq 0, Q_2 \geq 0
\end{align*}
\]

where \( Q_1 \) is the number of barrels of American pale ale contracted at brewery 1 and \( Q_2 \) is the number of barrels of American pale ale contracted at brewery 2. How many barrels should be contracted from each brewery to minimize the production cost \( C \)?

Solution To find the optimal production levels, we need to graph the feasible region and evaluate the corner points in the cost function. The borders of each inequality are given by the equations \( Q_1 + Q_2 = 10,000 \), \( Q_2 = 0.25Q_1 \), and \( Q_2 = Q_1 \). Each of these equations can be graphed by utilizing slope-intercept form, \( Q_2 = mQ_1 + b \). In this form, \( Q_1 \) is graphed horizontally, and \( Q_2 \) is graphed vertically.

<table>
<thead>
<tr>
<th>Equation</th>
<th>Slope-Intercept Form</th>
<th>Slope and Vertical Intercept</th>
</tr>
</thead>
</table>
| \( Q_1 + Q_2 = 10,000 \) | \( Q_2 = -Q_1 + 10,000 \) | slope = -1  
intercept = (0, 10,000) |
| \( Q_2 = 0.25Q_1 \) | \( Q_2 = 0.25Q_1 \) | slope = 0.25  
intercept = (0, 0) |
| \( Q_2 = Q_1 \) | \( Q_2 = Q_1 \) | slope = 1  
intercept = (0, 0) |

Utilizing this information, we can graph the borders.
The graph is shown in the first quadrant because of the non-negativity constraints $Q_1 \geq 0$ and $Q_2 \geq 0$.

To shade the solution for each inequality, we need to use a test point. Whenever possible we try to use $(0, 0)$ for simplicity. However, this point is on two of the lines so we need to use another point. Any point can be used, and in this example we’ll use $(4000, 0)$.

<table>
<thead>
<tr>
<th>Inequality</th>
<th>Set $Q_1 = 4000$ and $Q_2 = 0$</th>
<th>True or False?</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_1 + Q_2 \geq 10,000$</td>
<td>$4000 + 0 \geq 10,000$</td>
<td>False</td>
</tr>
<tr>
<td>$Q_2 \geq 0.25Q_1$</td>
<td>$0 \geq 0.25(4000)$</td>
<td>False</td>
</tr>
<tr>
<td>$Q_2 \leq Q_1$</td>
<td>$0 \leq 4000$</td>
<td>True</td>
</tr>
</tbody>
</table>

For the two inequalities that are false, we need to draw arrows on the lines pointing away from the test point. For the inequality that is true, we draw an arrow on the line pointing toward the test point. The arrows
indicate the half planes that match the solution set of the individual inequalities.

Figure 24 - The individual solutions to the inequalities in 0.

The solution sets for the individual inequalities in the system of inequalities overlap in the shaded region in Figure 25.
This feasible region is unbounded since it cannot be enclosed in a circle. If a minimum cost exists, it will occur at one or both of the two corner points of the region.

To find the corner points, we need to solve the systems of equations formed by the borders that intersect at the corner point. For instance, if we solve

\[ Q_1 + Q_2 = 10,000 \]
\[ Q_2 = Q_1 \]

we can locate the point at which the green and blue lines cross. This is easy to do if we replace \( Q_2 \) with \( Q_1 \) and solve for \( Q_1 \):

\[ Q_1 + Q_1 = 10,000 \]  \hspace{1cm} \text{Replace } Q_2 \text{ with } Q_1
\[ 2Q_1 = 10,000 \]  \hspace{1cm} \text{Combine like terms}
\[ Q_1 = 5000 \]  \hspace{1cm} \text{Divide both sides by 2}
Since $Q_2 = Q_1$, the corner point is at $(5000,5000)$.

The other corner point corresponds to the solution of the system

\[
Q_1 + Q_2 = 10,000 \\
Q_2 = 0.25Q_1.
\]

Replace $Q_2$ with $0.25Q_1$ and solve for $Q_1$:

\[
Q_1 + 0.25Q_1 = 10,000 \quad \text{(Replace } Q_2 \text{ with } 0.25Q_1) \\
1.25Q_1 = 10,000 \quad \text{(Combine like terms)} \\
Q_1 = 8000 \quad \text{(Divide both sides by } 1.25)
\]

Since $Q_2 = 0.25Q_1$, we can calculate $Q_2 = 0.25(8000) = 2000$ to yield the corner point $(8000,2000)$.

The optimal solution is found by evaluating the cost function at each of the corner points.

<table>
<thead>
<tr>
<th>Corner Point $(Q_1, Q_2)$</th>
<th>$C = 100Q_1 + 125Q_2$</th>
</tr>
</thead>
</table>
| $(5000, 5000)$          | $C = 100(5000) + 125(5000)$  
                          | $= 1,125,000$   |
| $(8000, 2000)$          | $C = 100(8000) + 125(2000)$  
                          | $= 1,050,000$   |
Figure 26 - The feasible region with the isocost line $z = 1,050,000$. Lower isocost lines are to the left of this line and outside of the feasible region.

The lowest cost is $1,050,000. It occurs when 8000 barrels from brewery 1 and 2000 barrels from brewery 2 are contracted.