4.4 The Simplex Method and the Standard Minimization Problem

Question 1: What is a standard minimization problem?

Question 2: How is the standard minimization problem related to the dual standard maximization problem?

Question 3: How do you apply the Simplex Method to a standard minimization problem?

In Section 4.3, the Simplex Method was used to solve the standard maximization problem. With some modifications, it can also be used to solve the standard minimization problem. These problems share characteristics and are called the dual of the other. In this section, we learn what a standard minimization problem is and how it is connected to the standard maximization problem. Utilizing the connection between the dual problems, we will solve the standard minimization problem with the Simplex Method.

Question 1: What is a standard minimization problem?

In Section 4.3, we learned that some types of linear programming problems, where the objective function is maximized, are called standard maximization problems. A similar form exists for another for linear programming problems where the objective function is minimized.

A standard minimization problem is a type of linear programming problem in which the objective function is to be minimized and has the form

$$w = d_1 y_1 + d_2 y_2 + \cdots + d_n y_n$$

where $d_1, \ldots, d_n$ are real numbers and $y_1, \ldots, y_n$ are decision variables. The decision variables must represent non-negative values. The other constraints for the standard minimization problem have the form

$$e_1 y_1 + e_2 y_2 + \cdots + e_n y_n \geq f$$

where $e_1, \ldots, e_n$ and f are real numbers and $f \geq 0$.

The standard minimization problem is written with the decision variables $y_1, \ldots, y_n$, but any letters could be used as long as the standard minimization problem and the corresponding dual maximization problem do not share the same variable names.

Often a problem can be rewritten to put it into standard minimization form. In particular, constraints are often manipulated algebraically so the each constraint has the form $e_1 y_1 + e_2 y_2 + \cdots + e_n y_n \geq f$. Example 1 demonstrates how a constraint can be changed to put it in the proper form.

For the problems in this section, we will require the coefficients of the objective function be positive. Although this is not a requirement of the Simplex Method, it simplifies the presentation in this section.

Example 1 Write As A Standard Minimization Problem

In section 4.2, we solved the linear programming problem

$$\text{Minimize } w = 4y_1 + y_2$$

subject to

$$y_2 \geq -\tfrac{1}{4}y_1 + 2$$

$$7y_1 + 4y_2 \geq 32$$

$$y_1 \geq 0, y_2 \geq 0$$

using a graph. Rewrite this linear programming problem as a standard minimization problem.

Solution In a standard minimization problem, the objective function must have the form $w = d_1y_1 + d_2y_2 + \cdots d_ny_n$ where $d_1, \ldots, d_n$ are real number constants and $y_1, \ldots, y_n$ are the decision variables. The objective function matches this form with $n = 2$.

Each constraint must have the form $e_1y_1 + e_2y_2 + \cdots + e_ny_n \geq f$ where $e_1, \ldots, e_n$ and f are real number constants. Additionally, the constant f must be non-negative. The second constraint, $7y_1 + 4y_2 \geq 32$, fits this form perfectly.

The first constraint appears to have the correct type of terms, but variable terms are on both sides of the inequality. To put in the proper format, add $\tfrac{1}{4}y_1$ to both sides of the inequality:

$$\tfrac{1}{4}y_1 + y_2 \geq 2$$

With this change, we can write the problem as a standard minimization problem,

$$\text{Minimize } w = 4y_1 + y_2$$

subject to

$$\tfrac{1}{4}y_1 + y_2 \geq 2$$

$$7y_1 + 4y_2 \geq 32$$

$$y_1 \geq 0, y_2 \geq 0$$

In addition to adding and subtracting terms to a constraint, we can also multiply or divide the terms in a constraint by nonzero real numbers. However, remember that the direction of the inequality changes when you multiply or divide by a negative number. This can complicate or even prevent a linear programming problem from being changed to standard minimization form.

Question 2: How is the standard minimization problem related to the dual standard maximization problem?

At this point, the connection between the standard minimization problem and the standard maximization problem is not clear. Let's look at an example of a standard minimization problem and another related standard maximization problem.

The linear programming problem

$$\text{Minimize } w = 10y_1 + 20y_2$$

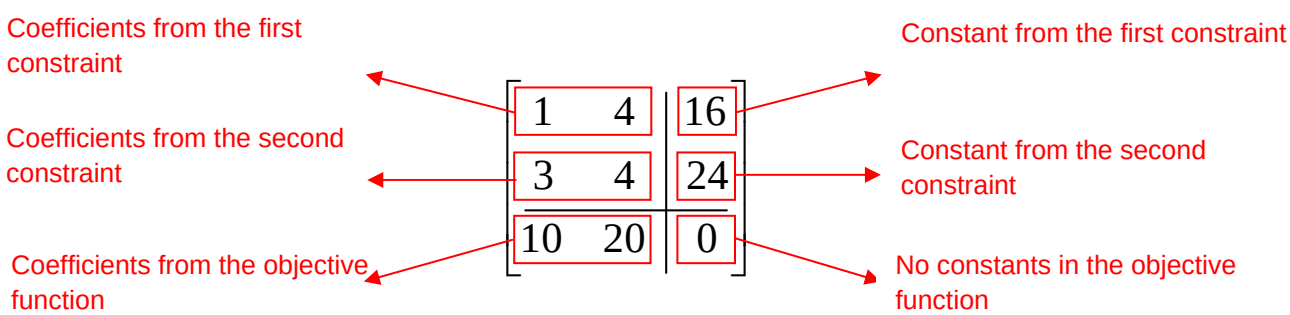
$$\text{subject to}$$

$$y_1 + 4y_2 \geq 16$$

$$3y_1 + 4y_2 \geq 24$$

$$y_1 \geq 0, y_2 \geq 0$$

is a standard minimization problem. The related dual maximization problem is found by forming a matrix before the objective function is modified or slack variables are added to the constraints. The entries in this matrix are formed from the coefficients and constants in the constraints and objective function:



To find the coefficients and constants in the dual problem, switch the rows and columns. In other words, make the rows in the matrix above become the columns in a new matrix,

$$\left[\begin{array}{cc|c} 1 & 3 & 10 \\ 4 & 4 & 20 \\ \hline 16 & 20 & 0 \end{array}\right]$$

The values in the new matrix help us to form the constraints and objective function in a standard maximization problem:

$$\left[\begin{array}{cc|c} 1 & 3 & 10 \\ 4 & 4 & 20 \\ \hline 16 & 24 & 0 \end{array}\right]$$

$$x_1 + 3x_2 \le 10$$

$$4x_1 + 4x_2 \le 20$$

$$\text{Maximize } z = 16x_1 + 24x_2$$

Notice the inequalities have switched directions since the dual problem is a standard maximization problem and the names of the variables are different from the original minimization problem. Putting these details together with non-negativity constraints, we get the standard maximization problem

$$\text{Maximize } z = 16x_1 + 24x_2$$

subject to

$$x_1 + 3x_2 \le 10$$

$$4x_1 + 4x_2 \le 20$$

$$x_1 \ge 0, x_2 \ge 0$$

This strategy works in general to find the dual problem.

Example 2 Find the Dual Maximization Problem

In Example 1, we rewrote a linear programming problem as a standard minimization problem,

$$\text{Minimize } w = 4y_1 + y_2$$

subject to

$$\tfrac{1}{4}y_1 + y_2 \geq 2$$

$$7y_1 + 4y_2 \geq 32$$

$$y_1 \geq 0, y_2 \geq 0$$

Find the dual maximization problem associated with this standard minimization problem.

Solution The dual maximization problem can be formed by examining a matrix where the first two rows are the coefficients and constants of the constraints and the last row contains the coefficients on the right side of the objective function.

In the case of this standard maximization problem, we get the 3 x 3 matrix

$$\left[\begin{array}{cc|c} \frac{1}{4} & 1 & 2 \\ 7 & 4 & 32 \\ \hline 4 & 1 & 0 \end{array}\right]$$

The vertical line separates the coefficients from the constants, and the horizontal line separates the entries corresponding to the constraints from the entries corresponding to the objective function. Notice that the entries are written before introducing slack variables or rearranging the objective function. The zero in the last column corresponding to the objective function comes from the fact that the objective function has no constants in it.

The coefficients and constants for the dual maximization problem are formed when the rows and columns of this matrix are interchanged. The new matrix,

$$\left[\begin{array}{cc|c} \frac{1}{4} & 7 & 4 \\ 1 & 4 & 1 \\ \hline 2 & 32 & 0 \end{array}\right]$$

is utilized to find the dual problem. The first row corresponds to the constraint $\frac{1}{4}x_1 + 7x_2 \le 4$. The second row corresponds to the constraint $x_1 + 4x_2 \le 1$. Notice that each constraint includes a less than or equal to ($\le$) to insure it fits the format of a standard maximization problem. The last row corresponds to the objective function $z = 2x_1 + 32x_2$.

These inequalities and equations are combined to yield the standard maximization problem

$$\text{Maximize z} = 2x_1 + 32x_2$$

subject to

$$\frac{1}{4}x_1 + 7x_2 \le 4$$

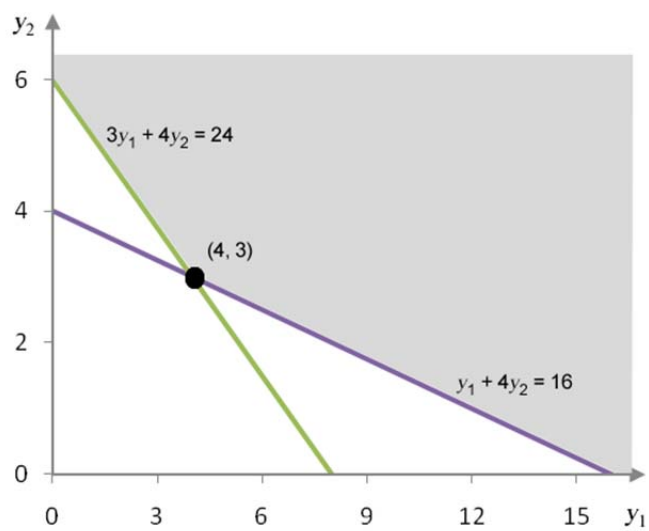
$$x_1 + 4x_2 \le 1$$

$$x_1 \ge 0, x_2 \ge 0$$

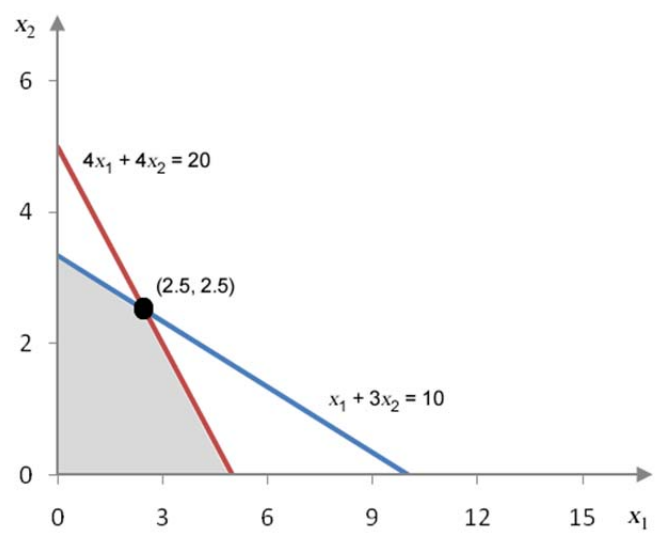
Question 3: How do you apply the Simplex Method to a standard minimization problem?

Example 2 illustrates how to convert a standard minimization problem into a standard maximization problem. These problems are called the dual of each other. The solutions of the dual problems are related and can be exploited to solve both problems simultaneously.

Let's look at the solution of each linear programming problem graphically. For each problem, let's look at a graph of the feasible region and a table of corner points with corresponding objective function values. From the table, we see that the solutions share the same objective function value at their respective solutions.



(y_1, y_2)	Minimize $w = 10y_1 + 20y_2$
$(16, 0)$	160
$(4, 3)$	100
$(0, 6)$	120



(x_1, x_2)	Maximize $z = 16x_1 + 24x_2$
$(5, 0)$	90
$(2.5, 2.5)$	100
$(0, \frac{10}{3})$	80

Although the corner points yielding the maximum or minimum are not the same, the value of the objective function at the optimal corner point is the same, 100. In other words,

$$w = 10(4) + 20(3) = 100$$

$$\text{yields the same value as}$$

$$z = 16(2.5) + 24(2.5) = 100$$

Another connection between the dual problems is evident if we apply the Simplex Method to the dual maximization problem

$$\text{Maximize } z = 16x_1 + 24x_2$$

$$\text{subject to}$$

$$x_1 + 3x_2 \le 10$$

$$4x_1 + 4x_2 \le 20$$

$$x_1 \ge 0, x_2 \ge 0$$

If we rearrange the objective function and add slack variables to the constraints, we get the system of equations

$$\begin{aligned} x_1 + 3x_2 + s_1 \qquad\qquad &= 10 \\ 4x_1 + 4x_2 \qquad + s_2 \qquad &= 20 \\ -16x_1 - 24x_2 \qquad\qquad + z &= 0 \end{aligned}$$

This system corresponds to the initial simplex tableau shown below. The pivot column is the second column and the quotients can be formed to yield

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & z & \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 3 & 1 & 0 & 0 & 10 \\ 4 & 4 & 0 & 1 & 0 & 20 \\ \hline -16 & -24 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \quad \begin{array}{l} \frac{10}{3} \approx 3.3 \\ \frac{20}{4} = 5 \\ \\ \end{array}$$

The pivot for this tableau is the 3 in the first row, second column.

If we multiply the first row by $\frac{1}{3}$, the pivot becomes a one and results in the tableau

$$\begin{array}{ccccc|c} 1 & 3 & 1 & 0 & 0 & 10 \\ & & & & \times & \frac{1}{3} \\ \hline \frac{1}{3} & 1 & \frac{1}{3} & 0 & 0 & \frac{10}{3} \end{array} \quad \boxed{\frac{1}{3}R_1 \text{ becomes } R_1} \quad \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{ccccc|c} \frac{1}{3} & 1 & \frac{1}{3} & 0 & 0 & \frac{10}{3} \\ 4 & 4 & 0 & 1 & 0 & 20 \\ \hline -16 & -24 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

The first simplex iteration is completed by creating zeros in the rest of the pivot column.

To change these entries, multiply the first row by -4 and add it to the second row. Then multiply the first row by 24 and add it to the third row.

$$\begin{array}{r} -4R_1: \\ +R_2: \\ \\ \end{array} \begin{array}{ccccc|c} -\frac{4}{3} & -4 & -\frac{4}{3} & 0 & 0 & -\frac{40}{3} \\ 4 & 4 & 0 & 1 & 0 & 20 \\ \hline \frac{8}{3} & 0 & -\frac{4}{3} & 1 & 0 & \frac{20}{3} \end{array} \quad \boxed{-4R_1 + R_2 \text{ becomes } R_2}$$

$$\begin{array}{r} 24R_1: \\ +R_3: \\ \\ \end{array} \begin{array}{ccccc|c} 8 & 24 & 8 & 0 & 0 & 80 \\ -16 & -24 & 0 & 0 & 1 & 0 \\ \hline -8 & 0 & 8 & 0 & 1 & 80 \end{array} \quad \boxed{24R_1 + R_3 \text{ becomes } R_3}$$

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{ccccc|c} \frac{1}{3} & 1 & \frac{1}{3} & 0 & 0 & \frac{10}{3} \\ \frac{8}{3} & 0 & -\frac{4}{3} & 1 & 0 & \frac{20}{3} \\ \hline -8 & 0 & 8 & 0 & 1 & 80 \end{array}\right] \end{array}$$

Now that the pivot is a one and the rest of the pivot column are zeros, look at the indicator row to see if another Simplex Method iteration is needed. Since the entry in the first column of the indicator row, -8, is negative, we make the first column the new pivot column.

The quotients for each row of the tableau are formed below:

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{ccccc|c} \frac{1}{3} & 1 & \frac{1}{3} & 0 & 0 & \frac{10}{3} \\ \frac{8}{3} & 0 & -\frac{4}{3} & 1 & 0 & \frac{20}{3} \\ \hline -8 & 0 & 8 & 0 & 1 & 80 \end{array}\right] \end{array} \qquad \begin{array}{l} \frac{10}{3} \div \frac{1}{3} = 10 \\ \frac{20}{3} \div \frac{8}{3} = 2.5 \end{array}$$

The smallest ratio is in the second row. The pivot, $\frac{8}{3}$, must be changed to a one by multiplying the second row by $\frac{3}{8}$,

Once the pivot is a one, row operations are used to change the rest of the pivot column to zeros.

The entry in the first row of the pivot column is

$$\begin{array}{ccccc|c} \frac{8}{3} & 0 & -\frac{4}{3} & 1 & 0 & \frac{20}{3} \\ & & & & \times & \frac{3}{8} \\ \hline 1 & 0 & -\frac{1}{2} & \frac{3}{8} & 0 & \frac{5}{2} \end{array} \xrightarrow{\boxed{\frac{3}{8}R_2 \text{ becomes } R_2}} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{ccccc|c} \frac{1}{3} & 1 & \frac{1}{3} & 0 & 0 & \frac{10}{3} \\ 1 & 0 & -\frac{1}{2} & \frac{3}{8} & 0 & \frac{5}{2} \\ \hline -8 & 0 & 8 & 0 & 1 & 80 \end{array}\right] \end{array}$$

changed to a zero by placing the sum of $-\frac{1}{3}$ times the second row and the first row in the first row. The entry in the third row of the pivot column is changed to a zero by placing the sum of 8 times the second row and the third row in the third row,

$$\begin{array}{r} -\frac{1}{3}R_2: \\ +R_1: \\ \\ \end{array} \begin{array}{ccccc|c} -\frac{1}{3} & 0 & \frac{1}{6} & -\frac{1}{8} & 0 & -\frac{5}{6} \\ \frac{1}{3} & 1 & \frac{1}{3} & 0 & 0 & \frac{10}{3} \\ \hline 0 & 1 & \frac{1}{2} & -\frac{1}{8} & 0 & \frac{5}{2} \end{array} \xrightarrow{\boxed{-\frac{1}{3}R_2 + R_1 \text{ becomes } R_1}}$$

$$\begin{array}{r} 8R_2: \\ +R_3: \\ \\ \end{array} \begin{array}{ccccc|c} 8 & 0 & -4 & 3 & 0 & 20 \\ -8 & 0 & 8 & 0 & 1 & 80 \\ \hline 0 & 0 & 4 & 3 & 1 & 100 \end{array} \xrightarrow{\boxed{8R_2 + R_3 \text{ becomes } R_3}} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{ccccc|c} 0 & 1 & \frac{1}{2} & -\frac{1}{8} & 0 & \frac{5}{2} \\ 1 & 0 & -\frac{1}{2} & \frac{3}{8} & 0 & \frac{5}{2} \\ \hline 0 & 0 & 4 & 3 & 1 & 100 \end{array}\right] \end{array}$$

Since the indicator row no longer contains any negative entries, we have reached the final tableau. If we examine the final simplex tableau carefully, we can see the solution to the standard maximization problem and the standard minimization problem:

$$
\begin{array}{c}
\begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & z & \end{array} \\
\left[\begin{array}{ccccc|c}
0 & 1 & \frac{1}{2} & -\frac{1}{8} & 0 & \frac{5}{2} \\
1 & 0 & -\frac{1}{2} & \frac{3}{8} & 0 & \frac{5}{2} \\
\hline
0 & 0 & 4 & 3 & 1 & 100
\end{array}\right]
\end{array}
$$

The solution to the standard maximization problem is $(x_1, x_2) = (\frac{5}{2}, \frac{5}{2})$.

The value for the objective function in the dual problems is $z = w = 100$.

The solution to the dual minimization problem is $(y_1, y_2) = (4, 3)$.

The final simplex tableau gives the solution to the standard maximization problem and the solution to the corresponding dual standard minimization problem. This means that as long as we can solve the standard maximization problem with the Simplex Method, we get the solution to the dual standard minimization problem for free. This suggests a strategy for solving standard minimization problems.

How to Solve a Standard Minimization Problem with the Dual Problem

1. Make sure the minimization problem is in standard form. If it is not in standard form, modify the problem to put it in standard form.

2. Find the dual standard maximization problem.

3. Apply the Simplex Method to solve the dual maximization problem.

4. Once the final simplex tableau has been calculated, the minimum value of the standard minimization problem's objective function is the same as the maximum value of the standard maximization problem's objective function.

5. The solution to the standard minimization problem is found in the bottom row of the final simplex tableau in the columns corresponding to the slack variables.

Example 3 Find the Optimal Solution

In section 4.2, we solved the linear programming problem

$$\text{Minimize } w = 4y_1 + y_2$$

subject to

$$y_2 \geq -\tfrac{1}{4}y_1 + 2$$

$$7y_1 + 4y_2 \geq 32$$

$$y_1 \geq 0, y_2 \geq 0$$

using a graph. In 1.1Question 1Example 2, we found the associated dual maximization problem,

$$\text{Maximize z} = 2x_1 + 32x_2$$

subject to

$$\tfrac{1}{4}x_1 + 7x_2 \leq 4$$

$$x_1 + 4x_2 \leq 1$$

$$x_1 \geq 0, x_2 \geq 0$$

Apply the Simplex Method to this dual problem to solve the minimization problem.

Solution In Example 1 and Example 2 we wrote this problem as a standard minimization problem and found the dual maximization problem. In this example, we'll take the dual problem,

$$\text{Maximize } z = 2x_1 + 32x_2$$

subject to

$$\tfrac{1}{4}x_1 + 7x_2 \le 4$$

$$x_1 + 4x_2 \le 1$$

$$x_1 \ge 0, x_2 \ge 0$$

and apply the Simplex Method.

The initial simplex tableau is formed from the system of equations

$$\begin{aligned} \tfrac{1}{4}x_1 + \ \ 7x_2 + s_1 \qquad\qquad &= 4 \\ x_1 + \ \ 4x_2 \qquad + s_2 \qquad &= 1 \\ -2x_1 - 32x_2 \qquad\qquad + z &= 0 \end{aligned}$$

Notice that the slack variables s_1 and s_2 are included in the equations corresponding to the constraints, and the objective function has been rearranged appropriately.

The initial tableau is

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & z & \end{array} \\ \left[\begin{array}{ccccc|c} \frac{1}{4} & 7 & 1 & 0 & 0 & 4 \\ 1 & 4 & 0 & 1 & 0 & 1 \\ \hline -2 & -32 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}.$$

The most negative entry in the indicator row is -32, so the second column is the pivot column. Now calculate the quotients to find the pivot row,

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & z & \end{array} \\ \left[\begin{array}{ccccc|c} \frac{1}{4} & 7 & 1 & 0 & 0 & 4 \\ 1 & 4 & 0 & 1 & 0 & 1 \\ \hline -2 & -32 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \quad \begin{array}{l} \frac{4}{7} \approx 0.57 \\ \frac{1}{4} = 0.25 \\ \end{array}$$

The smallest quotient corresponds to putting the pivot in the second row, second column. To change the entry in this position to a one, multiply the second row by $\frac{1}{4}$:

$$\begin{array}{ccccc|c} 1 & 4 & 0 & 1 & 0 & 1 \\ & & & & \times & \frac{1}{4} \\ \hline \frac{1}{4} & 1 & 0 & \frac{1}{4} & 0 & \frac{1}{4} \end{array} \xrightarrow{\frac{1}{4}R_2 \text{ becomes } R_2} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{ccccc|c} \frac{1}{4} & 7 & 1 & 0 & 0 & 4 \\ \frac{1}{4} & 1 & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ \hline -2 & -32 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

To put zeros in the rest of the pivot column, we utilize more row operations.

$$\begin{array}{lccccc|c} -7R_2: & -\frac{7}{4} & -7 & 0 & -\frac{7}{4} & 0 & -\frac{7}{4} \\ +R_1: & \frac{1}{4} & 7 & 1 & 0 & 0 & 4 \\ \hline & -\frac{6}{4} & 0 & 1 & -\frac{7}{4} & 0 & \frac{9}{4} \end{array} \xrightarrow{-7R_2+R_1 \text{ becomes } R_1} \begin{array}{c} \begin{array}{ccccc} x & y & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{ccccc|c} -\frac{6}{4} & 0 & 1 & -\frac{7}{4} & 0 & \frac{9}{4} \\ \frac{1}{4} & 1 & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ \hline 6 & 0 & 0 & 8 & 1 & 8 \end{array}\right] \end{array}$$

$$\begin{array}{lccccc|c} 32R_2: & 8 & 32 & 0 & 8 & 0 & 8 \\ +R_3: & -2 & -32 & 0 & 0 & 1 & 0 \\ \hline & 6 & 0 & 0 & 8 & 1 & 8 \end{array} \xrightarrow{32R_2+R_3 \text{ becomes } R_3}$$

Since the indicator row is non-negative, this tableau corresponds to the optimal solution. The solution is found in the indicator row under the columns for the slack variables. The lowest value for w is 8 and occurs at $(y_1, y_2) = (0, 8)$.

This strategy works for standard minimization problems involving more variables or more constraints. Example 4 has two decision variables, but three constraints. This changes the sizes of the matrices involved, but not the process of applying the Simplex Method to the dual standard maximization problem.

Example 4 Find the Minimum Cost

In Section 4.2, we found the cost C of contracting Q_1 barrels of American ale from contract brewery 1 and Q_2 barrels of America ale from contract brewery 2. The linear programming problem for this application is

$$\text{Minimize } C = 100Q_1 + 125Q_2$$

subject to

$$Q_1 + Q_2 \geq 10,000$$

$$Q_2 \geq 0.25Q_1$$

$$Q_2 \leq Q_1$$

$$Q_1 \geq 0, Q_2 \geq 0$$

Follow the parts a through c to solve this linear programming problem.

a. Rewrite this problem so that it is a standard minimization problem.

Solution The objective function must have the form $w = d_1y_1 + d_2y_2 + \cdots d_ny_n$ where $y_1, \ldots, y_n$ are the decision variables, and $d_1, \ldots, d_n$ are constants. In this case the decision variables are Q_1 and Q_2, and C is used instead of w. A different name for the variable is acceptable as long as the terms on the right side each contain a constant times a variable.

The constraints must have the form $e_1y_1 + e_2y_2 + \cdots + e_ny_n \geq f$, where $e_1, \ldots, e_n$ and f are constants. The first constraint, $Q_1 + Q_2 \geq 10,000$, has the proper format, but with the decision variables Q_1 and Q_2 instead of y_1 and y_2.

The second and third constraints must be modified to match the form $e_1 y_1 + e_2 y_2 + \cdots + e_n y_n \geq f$. Subtract $0.25Q_1$ from both sides of the constraint $Q_2 \geq 0.25Q_1$ to yield

$$-0.25Q_1 + Q_2 \geq 0$$

The third constraint is converted to the proper form by rearranging the inequality $Q_1 \geq Q_2$. Subtract Q_2 from both sides to yield

$$Q_1 - Q_2 \geq 0$$

These changes lead to a standard minimization problem,

$$\begin{aligned}
&\text{Minimize } C = 100Q_1 + 125Q_2 \\
&\text{subject to} \\
&\qquad Q_1 + Q_2 \geq 10,000 \\
&\qquad -0.25Q_1 + Q_2 \geq 0 \\
&\qquad Q_1 - Q_2 \geq 0 \\
&\qquad Q_1 \geq 0, Q_2 \geq 0
\end{aligned}$$

b. Find the dual problem for the standard minimization problem.

Solution The dual maximization problem is found by forming a matrix from the constraints and objective function. The coefficients and constants in the constraints compose the first three rows. The coefficients from the objective function are placed the fourth row of the matrix.

$$\text{Minimize } C = 100Q_1 + 125Q_2$$

subject to

$$\begin{aligned} Q_1 + Q_2 &\geq 10{,}000 \\ -0.25Q_1 + Q_2 &\geq 0 \\ Q_1 - Q_2 &\geq 0 \\ Q_1 \geq 0,\ Q_2 &\geq 0 \end{aligned}$$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 10{,}000 \\ -0.25 & 1 & 0 \\ 1 & -1 & 0 \\ \hline 100 & 125 & 0 \end{array}\right]$$

The dual maximization problem's coefficients and constant are found by switching the rows and columns of the matrix

$$\left[\begin{array}{cc|c} 1 & 1 & 10{,}000 \\ -0.25 & 1 & 0 \\ 1 & -1 & 0 \\ \hline 100 & 125 & 0 \end{array}\right] \xrightarrow[\text{rows and columns}]{\text{switch}} \left[\begin{array}{ccc|c} 1 & -0.25 & 1 & 100 \\ 1 & 1 & -1 & 125 \\ \hline 10{,}000 & 0 & 0 & 0 \end{array}\right]$$

We'll use the decision variables x_1, x_2, and x_3 and write the corresponding dual maximization problem,

$$\text{Maximize } z = 10{,}000x_1$$

subject to

$$\begin{aligned} x_1 - 0.25x_2 + x_3 &\leq 100 \\ x_1 + x_2 - x_3 &\leq 125 \\ x_1 \geq 0,\ x_2 \geq 0,\ x_3 &\geq 0 \end{aligned}$$

c. Apply the Simplex Method to the dual problem to find the solution to the standard minimization problem.

Solution The standard minimization problem is solved by applying the Simplex Method to the dual maximization problem. The first step is to write out the system of equations we'll work with including the slack variables:

$$
\begin{aligned}
x_1 - 0.25x_2 + x_3 + s_1 \qquad\qquad &= 100 \\
x_1 + \quad x_2 - x_3 \quad + s_2 \quad &= 125 \\
-10{,}000x_1 \qquad\qquad\qquad\qquad + z &= 0
\end{aligned}
$$

This system of equations corresponds to the initial simplex tableau

$$
\begin{array}{c}
\begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & z \end{array} \\
\left[\begin{array}{cccccc|c}
1 & -0.25 & 1 & 1 & 0 & 0 & 100 \\
1 & 1 & -1 & 0 & 1 & 0 & 125 \\
\hline
-10{,}000 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right]
\end{array}
$$

The only negative number in the indicator row is -10,000, so the pivot column is the first column. We choose the pivot row by forming quotients from the last column and the pivot column:

$$
\begin{array}{c}
\begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & z \end{array} \\
\left[\begin{array}{cccccc|c}
1 & -0.25 & 1 & 1 & 0 & 0 & 100 \\
1 & 1 & -1 & 0 & 1 & 0 & 125 \\
\hline
-10{,}000 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right]
\end{array}
\begin{array}{l}
\frac{100}{1} = 100 \\
\frac{125}{1} = 125 \\

\end{array}
$$

The smallest quotient is 100 so the first row is the pivot row.

Conveniently, the pivot is already a one and we can use row operations to change the rest of the column to zeros.

$$
\begin{array}{r}
-1R_1: \\
+R_2: \\
\hline
\end{array}
\left[\begin{array}{cccccc|c}
-1 & 0.25 & -1 & -1 & 0 & 0 & -100 \\
1 & 1 & -1 & 0 & 1 & 0 & 125 \\
\hline
0 & 1.25 & -2 & -1 & 1 & 0 & 25
\end{array}\right]
\xrightarrow{\boxed{\begin{array}{c} -1R_1 + R_2 \\ \text{becomes} \\ R_2 \end{array}}}
\begin{array}{c}
\begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & z \end{array} \\
\left[\begin{array}{cccccc|c}
1 & -0.25 & 1 & 1 & 0 & 0 & 100 \\
0 & 1.25 & -2 & -1 & 1 & 0 & 25 \\
\hline
0 & -2500 & 10{,}000 & 10{,}000 & 0 & 1 & 1{,}000{,}000
\end{array}\right]
\end{array}
$$

$$
\begin{array}{r}
10{,}000R_1: \\
+R_3: \\
\hline
\end{array}
\left[\begin{array}{cccccc|c}
10{,}000 & -2500 & 10{,}000 & 10{,}000 & 0 & 0 & 1{,}000{,}000 \\
-10{,}000 & 0 & 0 & 0 & 0 & 1 & 0 \\
\hline
0 & -2500 & 10{,}000 & 10{,}000 & 0 & 1 & 1{,}000{,}000
\end{array}\right]
\xrightarrow{\boxed{\begin{array}{c} 10{,}000R_1 + R_3 \\ \text{becomes} \\ R_3 \end{array}}}
$$

The new matrix still has a negative number in the indicator row, so we must choose a new pivot. The second column is the pivot column. The second row is the pivot row since it contains the only admissible quotient, $\frac{25}{1.25}$.

The pivot, 1.25, is changed to a one by multiplying the second row by $\frac{1}{1.25}$:

$$\begin{array}{cccccc|c} 0 & 1.25 & -2 & -1 & 1 & 0 & 25 \\ & & & & & \times & \frac{1}{1.25} \\ \hline 0 & 1 & -1.6 & -0.8 & 0.8 & 0 & 20 \end{array} \xrightarrow{\frac{1}{1.25}R_2 \text{ becomes } R_2} \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{cccccc|c} 1 & -0.25 & 1 & 1 & 0 & 0 & 100 \\ 0 & 1 & -1.6 & -0.8 & 0.8 & 0 & 20 \\ \hline 0 & -2500 & 10,000 & 10,000 & 0 & 1 & 1,000,000 \end{array}\right] \end{array}$$

With a one in the pivot, we can use row operations to put zeros in the rest of the pivot column. Multiply the pivot row by 0.25 and add it to the first row to put a zero at the top of the pivot column. Multiply the pivot row by 2500 and add it to the third row to put a zero at the bottom of the pivot row:

$$\begin{array}{r|cccccc|c} 0.25R_2: & 0 & 0.25 & -0.4 & -0.2 & 0.2 & 0 & 5 \\ +R_1: & 1 & -0.25 & 1 & 1 & 0 & 0 & 100 \\ \hline & 1 & 0 & 0.6 & 0.8 & 0.2 & 0 & 105 \end{array} \xrightarrow{0.25R_2 + R_1 \text{ becomes } R_1} \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{cccccc|c} 1 & 0 & 0.6 & 0.8 & 0.2 & 0 & 105 \\ 0 & 1 & -1.6 & -0.8 & 0.8 & 0 & 20 \\ \hline 0 & 0 & 6000 & 8000 & 2000 & 1 & 1,050,000 \end{array}\right] \end{array}$$

$$\begin{array}{r|cccccc|c} 2500R_2: & 0 & 2500 & -4000 & -2000 & 2000 & 0 & 50,000 \\ +R_3: & 0 & -2500 & 10,000 & 10,000 & 0 & 1 & 1,000,000 \\ \hline & 0 & 0 & 6,000 & 8,000 & 2000 & 1 & 1,050,000 \end{array} \xrightarrow{2500R_2 + R_3 \text{ becomes } R_3}$$

No entry in the indicator row is negative, so we know that this tableau corresponds to the solution. The solution to the minimization problem lies in the indicator row in the columns corresponding to the slack variables,

$$
\begin{array}{c}
\begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & z \end{array} \\
\left[\begin{array}{cccccc|c}
1 & 0 & 0.6 & 0.8 & 0.2 & 0 & 105 \\
0 & 1 & -1.6 & -0.8 & 0.8 & 0 & 20 \\
\hline
0 & 0 & 6000 & \boxed{8000} & \boxed{2000} & 1 & \boxed{1,050,000}
\end{array}\right]
\end{array}
$$

The lowe Solution to the minimization problem curs when $(($ Lowest cost $), 2000)$.

This means the lowest cost occurs when 8000 barrels are contracted from brewery 1 and 2000 barrels are contracted from brewery 2.